Math 218D: Week 1 Discussion

Study Copy

August 26, 2021

Problem 1. Fill in the blanks below.

$$A = \begin{bmatrix} 3 & -4 & 2 & 8 & 0 \\ 1 & 8 & 1/9 & 0 & 1 \\ 2 & 4 & -1 & \pi & 2 \end{bmatrix} \qquad a_{23} = \underline{\qquad} \qquad \operatorname{Col}_2(A) \in \mathbb{R}^{\underline{\quad}} \qquad A^\intercal = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

Problem 2. Fill in the blanks below, assuming that S is *symmetric*.

$$S = \begin{bmatrix} 5 & -4 & \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & 19 & \underline{\qquad} & -1 \\ 11 & 2 & 8 & 3 \\ 9 & \underline{\qquad} & \underline{\qquad} & -10 \end{bmatrix} \qquad \operatorname{trace}(S) = \underline{\qquad}$$

Problem 3. By definition, a matrix S is *symmetric* if $\underline{\qquad\qquad\qquad}$.

Problem 4. Suppose that A is $n \times n$ and let $S = A + A^\intercal$. Prove that S is symmetric.

Hint. This proof can be quickly accomplished by filling in the blanks below.

$$\underline{\qquad\qquad\qquad} = \underline{\qquad\qquad\qquad} = \underline{\qquad\qquad\qquad} = \underline{\qquad\qquad\qquad} = \underline{\qquad\qquad\qquad}$$

Problem 5. Consider the matrix R given by

$$R = \begin{bmatrix} 1 & -3 & 0 & -9 & 5 \\ 0 & 0 & 1 & 14 & 9 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \operatorname{Col}_4(R) = \underline{\qquad} \operatorname{Col}_1(R) + \underline{\qquad} \operatorname{Col}_3(R)$$

(a) Fill in the blanks above to express the fourth column of R as a linear combination of the first and third columns of R.

(b) Can the fifth column of R be expressed as a linear combination of the first and third columns of R? Explain why or why not.

Problem 6. We write $\mathbb{R}^9 \xrightarrow{A} \mathbb{R}^{22}$ to indicate that A is a $\underline{\qquad} \times \underline{\qquad}$ matrix.

Problem 7. Suppose $\mathbb{R}^{13} \xrightarrow{M^\intercal} \mathbb{R}^{37}$. Then M is a $\underline{\qquad} \times \underline{\qquad}$ matrix.

Problem 8. Fill in the blanks in the two equations below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \underline{\qquad} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} + \underline{\qquad} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} + \underline{\qquad} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} \qquad \begin{bmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = 11 \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} - 42 \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}$$

Problem 9. Fill in the blanks in each equation below.

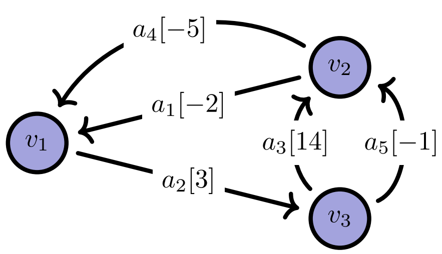
$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \text{the third column of } A \qquad \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \text{the first column minus the third column of } A$$

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \text{the sum of all columns of } A \qquad \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \text{twice the first column of } A$$

Problem 10. Suppose that A has four columns. Fill in the blanks in the equation on the right to validate the equation on the left.

$$\mathrm{Col}_1 + 3\ \mathrm{Col}_2 - 9\ \mathrm{Col}_3 = 6\ \mathrm{Col}_4 \qquad \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \boldsymbol{O}$$

Problem 11. Consider the weighted digraph G depicted below.



Use a matrix-vector product to calculate the weighted net flow through each node of G.

Math 218D: Week 2 Discussion

Study Copy

September 2, 2021

Problem 1. $\langle \begin{bmatrix} 1 & -3 & 0 & 2 \end{bmatrix}^\intercal, \begin{bmatrix} 2 & 1 & 5 & 0 \end{bmatrix}^\intercal \rangle =$ ____

Problem 2. Which of the following vectors is *orthogonal* to $\boldsymbol{v} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\intercal$?

○ $\boldsymbol{w} = \begin{bmatrix} 3 & -5 & 2 & 1 & 0 \end{bmatrix}^\intercal$ ○ $\boldsymbol{x} = \begin{bmatrix} 9 & 2 & 3 & -6 & -8 \end{bmatrix}^\intercal$ ○ $\boldsymbol{y} = \begin{bmatrix} -1 & -1 & 9 & -10 & 3 \end{bmatrix}^\intercal$

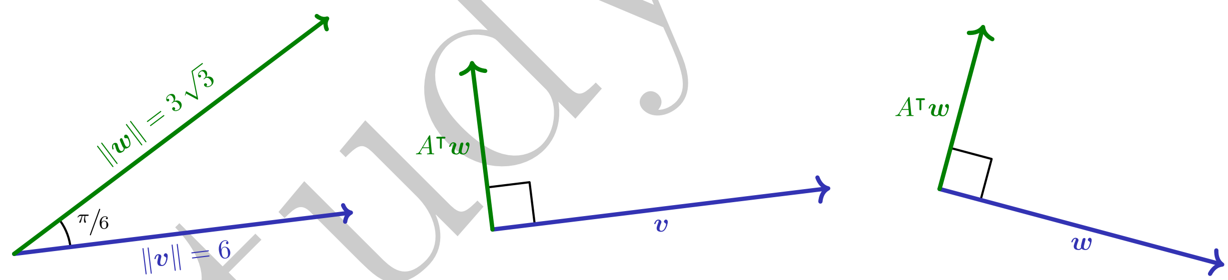
Problem 3. The length of $\boldsymbol{v}$ can be calculated with an inner product using the formula $\|\boldsymbol{v}\| =$ ____________.

Problem 4. The inner product can be interpreted geometrically with the formula $\langle \boldsymbol{v}, \boldsymbol{w} \rangle =$ ____________.

Problem 5. If we view $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ as $n \times 1$ matrices, then $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ can be calculated using matrix multiplication with the formula $\langle \boldsymbol{v}, \boldsymbol{w} \rangle =$ ____________.

Problem 6. The adjoint formula for inner products states that $\langle A\boldsymbol{v}, \boldsymbol{w} \rangle =$ ____________.

Problem 7. Suppose that A and B are matrices satisfying $A^\intercal B = I_n$ and that $\boldsymbol{v}$ and $\boldsymbol{w}$ vectors making the following diagrams accurate.



Calculate $\langle B\boldsymbol{v} - 3\,\boldsymbol{w}, 2\,A\boldsymbol{v} - A\boldsymbol{w} \rangle$.

Problem 8. One of the following calculations is possible and the other is not. Carry out the possible calculation.

$$\begin{bmatrix} 5 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 1 \\ 3 & -3 \end{bmatrix} = \underline{\hspace{6cm}} \qquad \begin{bmatrix} 5 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 3 \\ 1 & 1 & -3 \end{bmatrix} = \underline{\hspace{6cm}}$$

Problem 9. Fill in the blanks in each of the following two equations.

$$\begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \end{bmatrix} \begin{bmatrix} 0 & -3 & 1 & 0 \\ 1 & 9 & 0 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 50 & 5 & -4 \\ 12 & 193 & -3 & 19 \end{bmatrix} \qquad \begin{bmatrix} \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 2 & -5 \\ 0 & 9 \end{bmatrix}$$

Problem 10. Suppose that A and B are 2021×2021. Prove that $S = B^\intercal A + A^\intercal B$ is symmetric.

Problem 11. The last column of a matrix A is $\begin{bmatrix} 0 & 3 & 4 \end{bmatrix}^\intercal$ and the Gramian of A is

$$G = \begin{bmatrix} 9 & \underline{\quad} & -6 & \underline{\quad} \\ 3 & 14 & 13 & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & 29 & \underline{\quad} \\ 5 & 13 & 0 & \underline{\quad} \end{bmatrix} = \underline{\hspace{3cm}}$$

(a) Fill in the missing entries of G and fill in the formula used to calculate G.

(b) The number of rows of A is ____ and the number of columns of A is ____.

(c) Which (if any) of the columns of A is orthogonal to the third column of A?

Math 218D: Week 3 Discussion

Study Copy

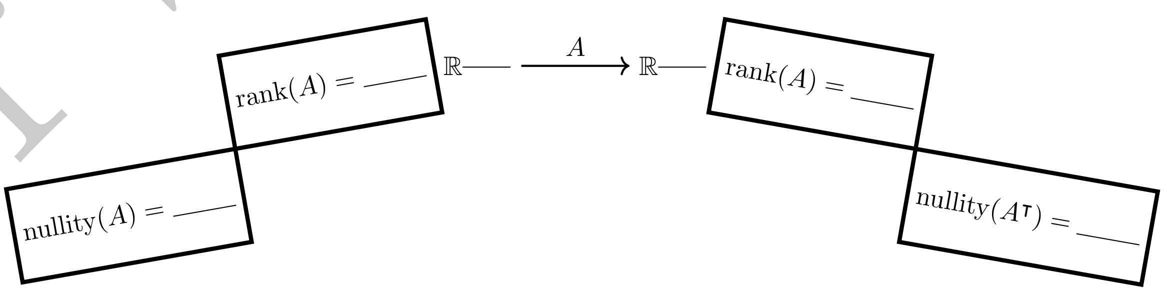
September 9, 2021

Problem 1. Consider the system of equations given by

$$\begin{array}{rcrcrcrcr} x_1 & + & 2\,x_2 & - & 4\,x_3 & + & 9\,x_4 & = & -2 \\ 5\,x_1 & + & 11\,x_2 & - & 13\,x_3 & + & 37\,x_4 & = & 7 \\ -\,3\,x_1 & - & 6\,x_2 & + & 12\,x_3 & - & 24\,x_4 & = & 0 \end{array}$$

Use the Gauß-Jordan algorithm to find the general solution to this system.

Problem 2. Suppose A is a matrix satisfying $\operatorname{rref}(A) = \begin{bmatrix} 1 & -13 & 0 & 6 \\ 0 & 0 & 1 & 17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Fill-in the blanks below.



Problem 3. Use the Gauß-Jordan algorithm to calculate $\operatorname{rref}(A)$ where $A = \begin{bmatrix} 3 & -6 & 12 & 0 & -9 \\ -7 & 14 & -28 & -5 & 26 \\ 5 & -12 & 12 & 2 & -13 \\ 2 & -3 & 12 & -3 & -5 \end{bmatrix}$.

Math 218D: Week 4 Discussion

Study Copy

September 16, 2021

Problem 1. Use the Gauß-Jordan algorithm to calculate $EA = R$ where $A = \begin{bmatrix} 1 & -5 & 0 & 3 \\ 0 & 0 & 5 & -5 \\ -5 & 25 & -11 & -4 \end{bmatrix}$.

Problem 2. Consider the $EA = R$ factorization and the vector $\boldsymbol{b}$ given by

$$\overset{E}{\begin{bmatrix} -6 & 5 & 2 & -13 \\ 4 & -4 & -2 & 9 \\ -9 & 9 & 4 & -21 \\ 1 & -2 & -1 & 3 \end{bmatrix}} \overset{A}{\begin{bmatrix} -3 & 21 & 2 & -26 & -5 & 3 \\ 3 & -21 & 1 & 14 & 38 & -2 \\ -3 & 21 & -3 & -6 & -60 & 1 \\ 2 & -14 & -1 & 16 & 7 & -2 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & -7 & 0 & 6 & 9 & 0 \\ 0 & 0 & 1 & -4 & 11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \qquad \boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Determine if $A\boldsymbol{x} = \boldsymbol{b}$ is consistent *without doing any row operations.*

Problem 3. Calculate $PA = LU$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & -2 \\ 3 & -1 & 1 \end{bmatrix}$.

Problem 4. Consider the $PA = LU$ factorization and the vector $\boldsymbol{b}$ given by

$$\overset{P}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}} \qquad \boldsymbol{b} = \begin{bmatrix} 15 \\ 11 \\ -15 \\ 13 \end{bmatrix}$$

Solve $A\boldsymbol{x} = \boldsymbol{b}$ *without doing any row reductions.*