Math 218D: Week 9 Discussion

Study Copy

October 21, 2021

Problem 1. Calculate $\begin{vmatrix} 1 & -9 & 3 & -8 \\ 9 & -81 & 27 & -70 \\ -5 & 45 & -14 & 29 \\ -7 & 60 & -16 & 52 \end{vmatrix}$.

Problem 2. Consider the following matrix factorization

$$\overset{P}{\begin{bmatrix} 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 0&-1&-1&1&6 \\ -1&-4&1&-1&-4 \\ -2&-10&0&2&-1 \\ 8&3&1&-2&-3 \\ 3&-1&0&1&1 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1&0&0&0&0 \\ -1/4&1&0&0&0 \\ -1/8&29/74&1&0&0 \\ 0&4/37&-1&1&0 \\ 3/8&17/74&-8/19&-12/19&1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 8&3&1&-2&-3 \\ 0&-37/4&1/4&3/2&-7/4 \\ 0&0&38/37&-68/37&-273/74 \\ 0&0&0&-1&5/2 \\ 0&0&0&0&97/38 \end{bmatrix}}$$

Calculate $\det(A)$.

Problem 3. For $n \times n$ matrices A and B, $\det(A^\intercal) =$ ____________ and $\det(AB) =$ ____________.

Problem 4. If possible, find 3×3 matrices A and B satisfying $\det(A + B) \neq \det(A) + \det(B)$. If this is not possible, then explain why.

Problem 5. The (i, j) *minor* of A is $M_{ij} =$ ____________ and the (i, j) *cofactor* is $C_{ij} =$ ____________.

Problem 6. Suppose that $\det(A) = 35$ and that each (i, j) minor of A is the (i, j) entry of $M = \begin{bmatrix} -45 & -9 & -41 & 5 \\ -10 & 5 & 15 & 5 \\ 45 & 16 & 34 & -5 \\ -10 & -2 & -13 & 5 \end{bmatrix}$.

(a) Find the cofactor matrix C of A and the adjugate matrix $\operatorname{adj}(A)$.

(b) Find three independent vectors orthogonal to the first column of A.

(c) Solve $A\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{b} = \begin{bmatrix} 0 & 7 & 0 & 0 \end{bmatrix}^\intercal$.

Problem 7. Suppose that λ is an eigenvalue of an $n \times n$ matrix A. Then $\det(\lambda \cdot I_n - A) =$ ______________.

Math 218D: Week 10 Discussion

Study Copy

October 28, 2021

Problem 1. The reciprocal of $z = 7 - 9\,i$ is ${}^1\!/_z =$ ____________ + ____________ i.

Problem 2. Consider the vectors $\boldsymbol{v} = \begin{bmatrix} 1+i & 5 \end{bmatrix}^\intercal$ and $\boldsymbol{w} = \begin{bmatrix} 1-3\,i & 2+i \end{bmatrix}^\intercal$ and the matrix $A = \left[\begin{smallmatrix} 2 & 1+i & -1 \\ 0 & 1 & 3-2\,i \end{smallmatrix}\right]$.

(a) Calculate $\|\boldsymbol{v}\|$.

(b) Calculate $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$.

(c) Calculate $A^*\boldsymbol{v}$.

Problem 3. We call a matrix A *Hermitian* if ____________. We call A *unitary* if ____________.

Problem 4. Suppose that H is Hermitian. Show that every diagonal entry of H is a real number.

Problem 5. Suppose that U is $n \times n$ unitary and that $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}^n$. Show that $\langle U\boldsymbol{v}, U\boldsymbol{w}\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle$.

Problem 6. The polynomial

$$f(t) = t^4 - 2\,t^3 - t^2 + t - 14$$

has four distinct roots r_1, r_2, r_3, and r_4.

(a) $r_1 + r_2 + r_3 + r_4 =$ ____________________ and $r_1 r_2 r_3 r_4 =$ ____________________

(b) Calculate $(1 - r_1)(1 - r_2)(1 - r_3)(1 - r_4)$.

Problem 7. Let r_1 and r_2 be the roots of

$$f(t) = -9\,t^2 - 2\,t - 1$$

Calculate $r_1^2 + r_2^2$.

Hint. Consider $(r_1 + r_2)^2$.

Math 218D: Week 11 Discussion

Name: ____

November 4, 2021

Problem 1. Consider the equation

$$\overset{A}{\begin{bmatrix} 18 & 10 & -5 & 0 \\ -10 & * & 8 & 5 \\ -6 & -28 & -14 & -14 \\ -11 & -30 & -11 & -10 \end{bmatrix}} = \overset{X}{\begin{bmatrix} 18 & 10 & -5 & 0 \\ -10 & * & 8 & 5 \\ -6 & -28 & -14 & -14 \\ -11 & -30 & -11 & -10 \end{bmatrix}} \overset{B}{\begin{bmatrix} -1 & * & -7 & -1 \\ 0 & 0 & * & * \\ -1 & 6 & -1 & -6 \\ * & * & 2 & -1 \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} 11 & 10 & 0 & 3 \\ * & -7 & * & -3 \\ -10 & * & * & -3 \\ 3 & 3 & * & * \end{bmatrix}}$$

where the entries marked $*$ are unknown. Find the missing entry of A.

Problem 2. Suppose that A has eigenspaces given by

$$\mathcal{E}_A(7) = \operatorname{Span}\{\begin{bmatrix}1 & 3 & 0\end{bmatrix}^\intercal\} \quad \mathcal{E}_A(1) = \operatorname{Span}\{\begin{bmatrix}-2 & -5 & -5\end{bmatrix}^\intercal\} \quad \mathcal{E}_A(-1) = \operatorname{Span}\{\begin{bmatrix}-3 & -7 & -9\end{bmatrix}^\intercal\}$$

Calculate $A^{2021}\boldsymbol{v}$ for $\boldsymbol{v} = \begin{bmatrix}0 & -1 & 3\end{bmatrix}^\intercal$.

Problem 3. Consider the factorization

$$\overset{A}{\begin{bmatrix} -233 & 693 \\ -84 & 250 \end{bmatrix}} = \begin{bmatrix} 3 & 11 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 19 \end{bmatrix} \begin{bmatrix} 3 & 11 \\ 1 & 4 \end{bmatrix}^{-1}$$

(a) $\det(A) =$ ______________

(b) Find the solution $\boldsymbol{u}(t)$ to the initial value problem $d\boldsymbol{u}/dt = A\boldsymbol{u}$ with $\boldsymbol{u}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^\intercal$.

(c) Let V be the vector space consisting of all vectors $\boldsymbol{v}$ such that the solution $\boldsymbol{u}(t)$ to $d\boldsymbol{u}/dt = A\boldsymbol{u}$ with $\boldsymbol{u}(0) = \boldsymbol{v}$ satisfies $\lim_{t \to \infty} \boldsymbol{u}(t) = \boldsymbol{O}$. Find a basis of V.

Math 218D: Week 12 Discussion

Study Copy

November 11, 2021

Problem 1. What is a spectral factorization? Which matrices have spectral factorizations?

Problem 2. Suppose S is a real-symmetric matrix whose eigenspaces are given by

$$\mathcal{E}_S(-3) = \operatorname{Span}\{\begin{bmatrix}1 & -2 & 0 & 2\end{bmatrix}^\intercal, \begin{bmatrix}-1 & -3 & -2 & 2\end{bmatrix}^\intercal\} \quad \mathcal{E}_S(5) = \operatorname{Span}\{\begin{bmatrix}0 & 2 & -1 & 2\end{bmatrix}^\intercal\} \quad \mathcal{E}_S(9) = ?$$

(a) Find a basis of $\mathcal{E}_S(9)$.

(b) Find a spectral factorization of S.

Problem 3. Show that the following quadratic form is *indefinite*.

$$q(\boldsymbol{x}) = 2\,x_1^2 + 8\,x_1x_2 + 2\,x_2^2 + 2\,x_1x_3 + 34\,x_2x_3 + 3670\,x_3^2 + 2\,x_1x_4 - 2\,x_2x_4 - 10\,x_3x_4 - 4\,x_4^2$$

Hint. We can do this very quickly by plugging in two vectors into $q(\boldsymbol{x})$.

Problem 4. Show that every Gramian is positive semidefinite. Under what condition is a Gramian positive definite?

Problem 5. Consider the spectral factorization

$$\overset{S}{\begin{bmatrix} 2 & 3 & -11 \\ 3 & 10 & -3 \\ -11 & -3 & 2 \end{bmatrix}} = \overset{U}{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}} \overset{D}{\begin{bmatrix} 16 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -9 \end{bmatrix}} \overset{U^*}{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}$$

Let $q(x_1, x_2, x_3)$ be the quadratic form corresponding to S.

(a) $q(x_1, x_2, x_3) = \underline{\qquad\qquad} \cdot x_1^2 + \underline{\qquad\qquad} \cdot x_2^2 + \underline{\qquad\qquad} \cdot x_3^2 + \underline{\qquad\qquad} \cdot x_1x_2 + \underline{\qquad\qquad} \cdot x_1x_3 + \underline{\qquad\qquad} \cdot x_2x_3$

(b) What is the definiteness of S? Select all that apply.

○ pos definite ○ pos semidefinite ○ neg definite ○ neg semidefinite ○ indefinite

(c) To "complete the square" we introduce variables $\{y_1, y_2, y_3\}$. Fill-in the blanks below to express $\{y_1, y_2, y_3\}$ in terms of the variables $\{x_1, x_2, x_3\}$.

$y_1 = \underline{\qquad\qquad\qquad\qquad}$ $y_2 = \underline{\qquad\qquad\qquad\qquad}$ $y_3 = \underline{\qquad\qquad\qquad\qquad}$

Fill-in the blanks below to express $q(x_1, x_2, x_3)$ as a function of $\{y_1, y_2, y_3\}$.

$q(y_1, y_2, y_3) = \underline{\qquad\qquad} \cdot y_1^2 + \underline{\qquad\qquad} \cdot y_2^2 + \underline{\qquad\qquad} \cdot y_3^2 + \underline{\qquad\qquad} \cdot y_1y_2 + \underline{\qquad\qquad} \cdot y_1y_3 + \underline{\qquad\qquad} \cdot y_2y_3$

(d) If possible, find a matrix A such that $S = A^*A$. If this is not possible, then explain why.