Duke University

Math 218

Matrices and Vector Spaces

Exam I

Name: Solutions

NetID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

September 24, 2021

- There are 100 points and 8 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

(5 pts) **Problem 1.** Fill in the blanks in the equations below.

$$\begin{bmatrix}3 & 1\\5 & 7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \underline{\ a\ }\begin{bmatrix}\underline{3}\\ \underline{5}\end{bmatrix} + \underline{\ b\ }\begin{bmatrix}\underline{1}\\ \underline{7}\end{bmatrix}$$

$$\begin{bmatrix}\underline{1} & \underline{4}\\ \underline{3} & \underline{5}\\ \underline{7} & \underline{2}\end{bmatrix}\begin{bmatrix}\underline{x}\\ \underline{y}\end{bmatrix} = x\begin{bmatrix}1\\3\\7\end{bmatrix} + y\begin{bmatrix}4\\5\\2\end{bmatrix}$$

$$\begin{bmatrix}6\,a_{11}+a_{12} & a_{11}\\ 6\,a_{21}+a_{22} & a_{21}\\ 6\,a_{31}+a_{32} & a_{31}\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{bmatrix}\begin{bmatrix}\underline{6} & \underline{1}\\ \underline{1} & \underline{0}\end{bmatrix}$$

$$\mathbb{R}^{\underline{4}} \xrightarrow{\left[\begin{smallmatrix}1&2&3&4\\5&6&7&8\end{smallmatrix}\right]} \mathbb{R}^{\underline{2}}$$

(10 pts) **Problem 2.** A matrix A is called *skew-symmetric* if $A^\intercal = -A$. Consider the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ given by

$$\boldsymbol{v} = \begin{bmatrix}2 & 1 & -2 & 0\end{bmatrix}^\intercal \qquad \boldsymbol{w} = \begin{bmatrix}-7 & 2 & 2 & 1\end{bmatrix}^\intercal$$

Suppose that A is a skew-symmetric matrix satisfying $A\boldsymbol{v} = \begin{bmatrix}-1 & 2 & 0 & -6\end{bmatrix}^\intercal$. Find $\langle \boldsymbol{v}, A\boldsymbol{w}\rangle$.

Solution. $\langle \boldsymbol{v}, A\boldsymbol{w}\rangle = \langle A^\intercal \boldsymbol{v}, \boldsymbol{w}\rangle = \langle -A\boldsymbol{v}, \boldsymbol{w}\rangle = \langle \begin{bmatrix}1 & -2 & 0 & 6\end{bmatrix}^\intercal, \begin{bmatrix}-7 & 2 & 2 & 1\end{bmatrix}^\intercal\rangle = -5$

(10 pts) **Problem 3.** Find a unit vector $\boldsymbol{u} \in \mathbb{R}^3$ orthogonal to $\begin{bmatrix}1 & 0 & 2\end{bmatrix}^\intercal$ and makes an angle of $\pi/4$ with $\begin{bmatrix}0 & 1 & 0\end{bmatrix}^\intercal$.
Hint. Recall that $\cos(\pi/4) = 1/\sqrt{2}$.

Solution. Write $\boldsymbol{u} = \begin{bmatrix}u_1 & u_2 & u_3\end{bmatrix}^\intercal$. The orthogonality condition requires $u_1 + 2\,u_3 = 0$ and the other angle gives $u_2 = 1/\sqrt{2}$.
Now, since $\boldsymbol{u}$ is a unit vector, we have

$$1 = u_1^2 + u_2^2 + u_3^2 = (-2\,u_3)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + u_3^2 = 5\,u_3^2 + \frac{1}{2}$$

This implies that $u_3 = \pm 1/\sqrt{10}$. Consequently, $u_1 = \mp 2/\sqrt{10}$. Our desired vector is then $\boldsymbol{u} = \begin{bmatrix}\mp 2/\sqrt{10} & 1/\sqrt{2} & \pm 1/\sqrt{10}\end{bmatrix}^\intercal$.

Problem 4. Suppose that c is a scalar and consider $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 2 \\ 0 & -2 & c^2-4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 3 \\ c \end{bmatrix}$.

(11 pts) (*a*) Use the Gauß-Jordan algorithm to find all values of c so that the system $A\boldsymbol{x} = \boldsymbol{b}$ has no solution, exactly one solution, or infinitely many solutions. Fill in the blanks below with your conditions.

Solution. Let's row-reduce the system $[A \mid \boldsymbol{b}]$.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 2 & 2 & 3 \\ 0 & -2 & c^2-4 & c \end{array}\right] \xrightarrow{\boldsymbol{r}_2 - \boldsymbol{r}_1 \to \boldsymbol{r}_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & -2 & c^2-4 & c \end{array}\right]$$

$$\xrightarrow{1/3 \cdot \boldsymbol{r}_2 \to \boldsymbol{r}_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & c^2-4 & c \end{array}\right]$$

$$\xrightarrow[\boldsymbol{r}_3 + 2\cdot\boldsymbol{r}_2 \to \boldsymbol{r}_3]{\boldsymbol{r}_1 + \boldsymbol{r}_2 \to \boldsymbol{r}_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & c^2-4 & c+2 \end{array}\right]$$

We find that inconsistency ocurs when $c^2 - 4 = 0$ but $c + 2 \neq 0$, which means $c = 2$. Consistency with one solution requires $c^2 - 4 \neq 0$, which means $c \neq \pm 2$. Consistency with infinitely many solutions requires $c^2 - 4 = 0$ and $c + 2 = 0$, which means $c = -2$.

no solutions: ____$c = 2$____ exactly one solution: ____$c \neq \pm 2$____ infinitely many solutions: ____$c = -2$____

(11 pts) (*b*) Consider the case $c = 3$. Find the solution to the system $A\boldsymbol{x} = \boldsymbol{b}$ and express $\boldsymbol{b}$ as a linear combination of the columns of A.

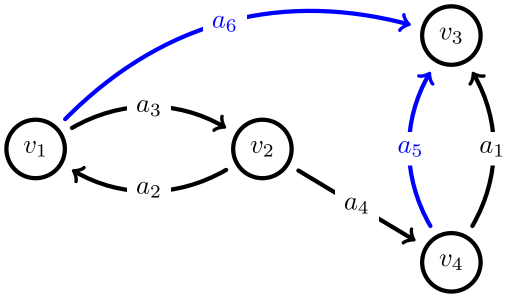
Solution. Continuing our row-reductions from before with $c = 3$ gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 5 & 5 \end{array}\right] \xrightarrow{1/5 \cdot \boldsymbol{r}_3 \to \boldsymbol{r}_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{\boldsymbol{r}_1 - 2\cdot\boldsymbol{r}_3 \to \boldsymbol{r}_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Our desired solution is evidently $\boldsymbol{x} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^{\intercal}$ and our linear combination is

$$\overset{\boldsymbol{b}}{\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}} = (-1) \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

(9 pts) **Problem 5.** The data below depicts a digraph G, the incidence matrix A of G, and $\operatorname{rref}(A)$.



$$\operatorname{rref}\overset{A}{\begin{bmatrix} 0 & 1 & -1 & 0 & \underline{\;0\;} & \underline{-1} \\ 0 & -1 & 1 & -1 & \underline{\;0\;} & \underline{\;0\;} \\ 1 & 0 & 0 & 0 & \underline{\;1\;} & \underline{\;1\;} \\ -1 & 0 & 0 & 1 & \underline{-1} & \underline{\;0\;} \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that G is missing two arrows. Draw these arrows and fill in the blanks in A. You can use the space below for scratch work, but you do not need to justify your answer.

Solution. We can figure out the missing columns of A by looking at the column relations given in $\operatorname{rref}(A)$.

$$\operatorname{Col}_5 = \operatorname{Col}_1 = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}^\intercal \qquad \operatorname{Col}_6 = \operatorname{Col}_1 - \operatorname{Col}_2 + \operatorname{Col}_4 = \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}^\intercal$$

Problem 6. Consider the $EA = R$ factorization and the vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$ given by

$$\overset{E}{\begin{bmatrix} -1 & 0 & 2 & -2 \\ 3 & -2 & 1 & 3 \\ -2 & 1 & 1 & -3 \\ 2 & -1 & 0 & 3 \end{bmatrix}} \overset{A}{\begin{bmatrix} -3 & 9 & 2 & 4 & 0 \\ -3 & 9 & 1 & 5 & 9 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -3 & -1 & -1 & 3 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & -3 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \qquad \boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{b}_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

(8 pts) (*a*) Find all solutions to each of the systems $A\boldsymbol{x} = \boldsymbol{b}_1$, $A\boldsymbol{x} = \boldsymbol{b}_2$, and $A\boldsymbol{x} = \boldsymbol{b}_3$. Write your solutions as a linear combination of vectors.

Solution. Recall that $[A \mid \boldsymbol{b}]$ reduces to $[R \mid E\boldsymbol{b}]$. Since R has one row of zeros at the bottom, we see that the consistency of $A\boldsymbol{x} = \boldsymbol{b}$ hinges on whether or not $\boldsymbol{b}$ is orthogonal to the last row of E. This is only true for $\boldsymbol{b}_3$. We then use $[R \mid E\boldsymbol{b}_3]$ to solve our system

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & 0 & -4 & -2 \\ 0 & 0 & 1 & 0 & -8 & -3 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \overset{\boldsymbol{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}} = \begin{bmatrix} 3\,c_1 + 4\,c_2 - 2 \\ c_1 \\ 8\,c_2 - 3 \\ -c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 0 \\ 8 \\ -1 \\ 1 \end{bmatrix}$$

(8 pts) (*b*) Find the third column of E^{-1}.

Solution. We know that $A = E^{-1}R$. The fourth column of R is $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\intercal$, so the third column of E^{-1} is the fourth column of A, which is $\begin{bmatrix} 4 & 5 & 1 & -1 \end{bmatrix}^\intercal$.

Problem 7. Consider the matrices P, L, and U and the vector $\boldsymbol{b}$ (whose last coordinate is t) given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ -2 & 2 & 5 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -3 & -1 & 4 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2 \\ 2 \\ 10 \\ t \end{bmatrix}$$

Suppose that A and B are matrices satisfying $PA = LU$ and $PB = L^{\intercal}L$.

(3 pts) (*a*) $\operatorname{rank}(A) =$ __2__, $\operatorname{nullity}(A) =$ __3__, and $\operatorname{nullity}(A^{\intercal}) =$ __2__

(2 pts) (*b*) In $A\boldsymbol{x} = \boldsymbol{O}$, which variables are *free*? ◯ x_1 √ x_2 ◯ x_3 √ x_4 √ x_5

(8 pts) (*c*) Find all values of t for which $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

Solution. As we saw in class, we can start by solving $L\boldsymbol{y} = P\boldsymbol{b}$.

$$\begin{array}{rcccccccl} y_1 & & & & & = & -2 & \to & y_1 = -2 \\ -3\cdot(-2) & + & y_2 & & & = & t & \to & y_2 = t-6 \\ -5\cdot(-2) & & & + & y_3 & = & 10 & \to & y_3 = 0 \\ -2\cdot(-2) & + & 2\cdot(t-6) & + & 5\cdot 0 + y_4 & = & 2 & \to & y_4 = -2\,t+10 \end{array}$$

This gives $\boldsymbol{y} = \begin{bmatrix} -2 & t-6 & 0 & -2\,t+10 \end{bmatrix}^{\intercal}$. The consistency of $A\boldsymbol{x} = \boldsymbol{b}$ now hinges on the consistency of $U\boldsymbol{x} = \boldsymbol{y}$, which requires $-2\,t+10 = 0$ so $t = 5$.

(7 pts) (*d*) Set $t = 3$ so $\boldsymbol{b} = \begin{bmatrix} -2 & 2 & 10 & 3 \end{bmatrix}^{\intercal}$. Find the solution to $B\boldsymbol{x} = \boldsymbol{b}$.

Solution. The equation $B\boldsymbol{x} = \boldsymbol{b}$ is equivalent to $PB\boldsymbol{x} = P\boldsymbol{b}$, which is $L^{\intercal}L\boldsymbol{x} = P\boldsymbol{b}$. If we put $\boldsymbol{y} = L\boldsymbol{x}$, then we can start with $L^{\intercal}\boldsymbol{y} = P\boldsymbol{b}$ (writing the equations in reverse order).

$$\begin{array}{rcccccccl} & & & & y_4 & = & 2 & \to & y_4 = 2 \\ & & & & y_3 + 5\cdot 2 & = & 10 & \to & y_3 = 0 \\ & & y_2 & + & 0\cdot 0 + 2\cdot 2 & = & 3 & \to & y_2 = -1 \\ y_1 & - & 3\,(-1) & - & 5\cdot 0 - 2\cdot 2 & = & -2 & \to & y_1 = -1 \end{array}$$

Then we have $L\boldsymbol{x} = \boldsymbol{y}$.

$$\begin{array}{rcccccccl} x_1 & & & & & = & -1 & \to & x_1 = -1 \\ -3\cdot(-1) & + & x_2 & & & = & -1 & \to & x_2 = -4 \\ -5\cdot(-1) & & & + & x_3 & = & 0 & \to & x_3 = -5 \\ -2\cdot(-1) & + & 2\cdot(-4) & + & 5\cdot(-5) + x_4 & = & -2 & \to & x_4 = 33 \end{array}$$

Our solution is thus $\boldsymbol{x} = \begin{bmatrix} -1 & -4 & -5 & 33 \end{bmatrix}^{\intercal}$.

(8 pts) **Problem 8.** Suppose that A is a matrix whose eigenvalues are $\text{E-Vals}(A) = \{-3, 5\}$ with geometric multiplicities given by $\operatorname{gm}_A(-3) = 7$ and $\operatorname{gm}_A(5) = 2$. Find all geometric multiplicities of all eigenvalues of $M = 6\,I - A$.

Hint. Start by looking at $\operatorname{nullity}(\lambda\,I - M)$.

Solution. We are given that

$$\operatorname{nullity}(-3\,I - A) = \operatorname{gm}_A(-3) = 7 \qquad \operatorname{nullity}(5\,I - A) = \operatorname{gm}_A(5) = 2$$

For any λ we then have

$$\operatorname{nullity}(\lambda\,I - M) = \operatorname{nullity}(\lambda\,I - (6\,I - A)) = \operatorname{nullity}((6-\lambda)\,I - A) = \begin{cases} 7 & \text{if } 6-\lambda = -3 \\ 2 & \text{if } 6-\lambda = 5 \\ 0 & \text{if } 6-\lambda \neq -3, 5 \end{cases} = \begin{cases} 7 & \text{if } \lambda = 9 \\ 2 & \text{if } \lambda = 1 \\ 0 & \text{if } \lambda \neq 9, 1 \end{cases}$$

This shows that $\text{E-Vals}(M) = \{9, 1\}$ with $\operatorname{gm}_M(9) = 7$ and $\operatorname{gm}_M(1) = 2$.