Duke University

Math 218

Matrices and Vector Spaces

Exam I

Name: ______________________ *NetID:* ______________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

September 24, 2021

- There are 100 points and 8 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke University MATH

(5 pts) **Problem 1.** Fill in the blanks in the equations below.

$$\begin{bmatrix} 3 & 1 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \underline{\qquad} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} + \underline{\qquad} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} \qquad \begin{bmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} + y \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 6\,a_{11} + a_{12} & a_{11} \\ 6\,a_{21} + a_{22} & a_{21} \\ 6\,a_{31} + a_{32} & a_{31} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{bmatrix} \qquad \mathbb{R}^{\underline{\qquad}} \xrightarrow{\left[\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{smallmatrix}\right]} \mathbb{R}^{\underline{\qquad}}$$

(10 pts) **Problem 2.** A matrix A is called *skew-symmetric* if $A^\intercal = -A$. Consider the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ given by

$$\boldsymbol{v} = \begin{bmatrix} 2 & 1 & -2 & 0 \end{bmatrix}^\intercal \qquad \boldsymbol{w} = \begin{bmatrix} -7 & 2 & 2 & 1 \end{bmatrix}^\intercal$$

Suppose that A is a skew-symmetric matrix satisfying $A\boldsymbol{v} = \begin{bmatrix} -1 & 2 & 0 & -6 \end{bmatrix}^\intercal$. Find $\langle \boldsymbol{v}, A\boldsymbol{w} \rangle$.

(10 pts) **Problem 3.** Find a unit vector $\boldsymbol{u} \in \mathbb{R}^3$ orthogonal to $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^\intercal$ and makes an angle of $\pi/4$ with $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\intercal$.

Hint. Recall that $\cos(\pi/4) = 1/\sqrt{2}$.

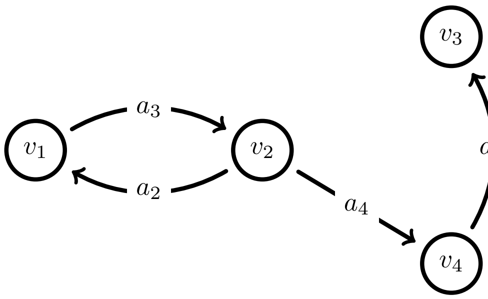
Problem 4. Suppose that c is a scalar and consider $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 2 \\ 0 & -2 & c^2 - 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 3 \\ c \end{bmatrix}$.

(11 pts) (a) Use the Gauß-Jordan algorithm to find all values of c so that the system $A\boldsymbol{x} = \boldsymbol{b}$ has no solution, exactly one solution, or infinitely many solutions. Fill in the blanks below with your conditions.

no solutions: ______________ exactly one solution: ______________ infinitely many solutions: ______________

(11 pts) (b) Consider the case $c = 3$. Find the solution to the system $A\boldsymbol{x} = \boldsymbol{b}$ and express $\boldsymbol{b}$ as a linear combination of the columns of A.

(9 pts) **Problem 5.** The data below depicts a digraph G, the incidence matrix A of G, and $\operatorname{rref}(A)$.



$$\operatorname{rref}\overset{A}{\begin{bmatrix} 0 & 1 & -1 & 0 & \_\_\_ & \_\_\_ \\ 0 & -1 & 1 & -1 & \_\_\_ & \_\_\_ \\ 1 & 0 & 0 & 0 & \_\_\_ & \_\_\_ \\ -1 & 0 & 0 & 1 & \_\_\_ & \_\_\_ \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that G is missing two arrows. Draw these arrows and fill in the blanks in A. You can use the space below for scratch work, but you do not need to justify your answer.

Problem 6. Consider the $EA = R$ factorization and the vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$ given by

$$\overset{E}{\begin{bmatrix} -1 & 0 & 2 & -2 \\ 3 & -2 & 1 & 3 \\ -2 & 1 & 1 & -3 \\ 2 & -1 & 0 & 3 \end{bmatrix}} \overset{A}{\begin{bmatrix} -3 & 9 & 2 & 4 & 0 \\ -3 & 9 & 1 & 5 & 9 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -3 & -1 & -1 & 3 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & -3 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \qquad \boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{b}_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

(8 pts) (a) Find all solutions to each of the systems $A\boldsymbol{x} = \boldsymbol{b}_1$, $A\boldsymbol{x} = \boldsymbol{b}_2$, and $A\boldsymbol{x} = \boldsymbol{b}_3$. Write your solutions as a linear combination of vectors.

(8 pts) (b) Find the third column of E^{-1}.

Problem 7. Consider the matrices P, L, and U and the vector $\boldsymbol{b}$ (whose last coordinate is t) given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ -2 & 2 & 5 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -3 & -1 & 4 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2 \\ 2 \\ 10 \\ t \end{bmatrix}$$

Suppose that A and B are matrices satisfying $PA = LU$ and $PB = L^{\intercal}L$.

(3 pts) (a) $\operatorname{rank}(A) =$ ________, $\operatorname{nullity}(A) =$ ________, and $\operatorname{nullity}(A^{\intercal}) =$ ________

(2 pts) (b) In $A\boldsymbol{x} = \boldsymbol{O}$, which variables are *free*? ○ x_1 ○ x_2 ○ x_3 ○ x_4 ○ x_5

(8 pts) (c) Find all values of t for which $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

(7 pts) (d) Set $t = 3$ so $\boldsymbol{b} = \begin{bmatrix} -2 & 2 & 10 & 3 \end{bmatrix}^{\intercal}$. Find the solution to $B\boldsymbol{x} = \boldsymbol{b}$.

(8 pts) **Problem 8.** Suppose that A is a matrix whose eigenvalues are $\text{E-Vals}(A) = \{-3, 5\}$ with geometric multiplicities given by $\operatorname{gm}_A(-3) = 7$ and $\operatorname{gm}_A(5) = 2$. Find all geometric multiplicities of all eigenvalues of $M = 6\,I - A$.

Hint. Start by looking at $\operatorname{nullity}(\lambda\,I - M)$.