Duke University

Math 218

Matrices and Vector Spaces

Exam II

Name: Solutions

NetID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

October 22, 2021

- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$.

(6 pts) (*a*) Determine if $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\intercal \in \operatorname{Null}(A)$.

Solution. This is a question of whether or not $A\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\intercal = \boldsymbol{O}$. The calculation

$$\overset{A}{\begin{bmatrix} 1 & 0 & -2 \\ -1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

then tells us that $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\intercal \notin \operatorname{Null}(A)$.

(6 pts) (*b*) Determine if $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\intercal \in \operatorname{Col}(A)$.

Solution. This is a question of whether or not $A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\intercal$ is consistent, which we determine with a few row-reductions

$$\left[\begin{array}{rrr|r} 1 & 0 & -2 & 1 \\ -1 & -1 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{array}\right] \xrightarrow[\boldsymbol{r}_3 - \boldsymbol{r}_1 \to \boldsymbol{r}_3]{\boldsymbol{r}_2 + \boldsymbol{r}_1 \to \boldsymbol{r}_2} \left[\begin{array}{rrr|r} 1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent, so $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\intercal \in \operatorname{Col}(A)$. Additionally, one might notice that $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\intercal$ is the first column of A minus the second column.

(6 pts) (*c*) Is $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\intercal$ an eigenvector of A? If so, what is the associated eigenvalue?

Solution. This is a question of whether or not $A\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\intercal = \lambda \cdot \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\intercal$ for some scalar λ. Here, we have

$$\overset{A}{\begin{bmatrix} 1 & 0 & -2 \\ -1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We find that $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\intercal$ is an eigenvector of A and the associated eigenvalue is $\lambda = -1$.

(7 pts) **Problem 2.** Suppose that A is a matrix and $\boldsymbol{v}$ is a vector satisfying $\boldsymbol{v} \in \mathcal{E}_A(-3)$. Show that $\boldsymbol{v}$ is an eigenvector of $M = A^2 - A$ and identify the corresponding eigenvalue.

Solution. We are given that $\boldsymbol{v} \in \mathcal{E}_A(-3)$ which means $A\boldsymbol{v} = -3 \cdot \boldsymbol{v}$. It follows that

$$M\boldsymbol{v} = (A^2 - A)\boldsymbol{v} = A^2\boldsymbol{v} - A\boldsymbol{v} = (-3)^2 \cdot \boldsymbol{v} - (-3) \cdot \boldsymbol{v} = (9 + 3) \cdot \boldsymbol{v} = 12 \cdot \boldsymbol{v}$$

This shows that $\boldsymbol{v} \in \mathcal{E}_M(12)$.

Problem 3. The first row of a matrix A is the vector $\begin{bmatrix}1 & -1 & 0 & 1\end{bmatrix}^\intercal$ and the first column is the vector $\begin{bmatrix}1 & 3 & -1\end{bmatrix}^\intercal$.

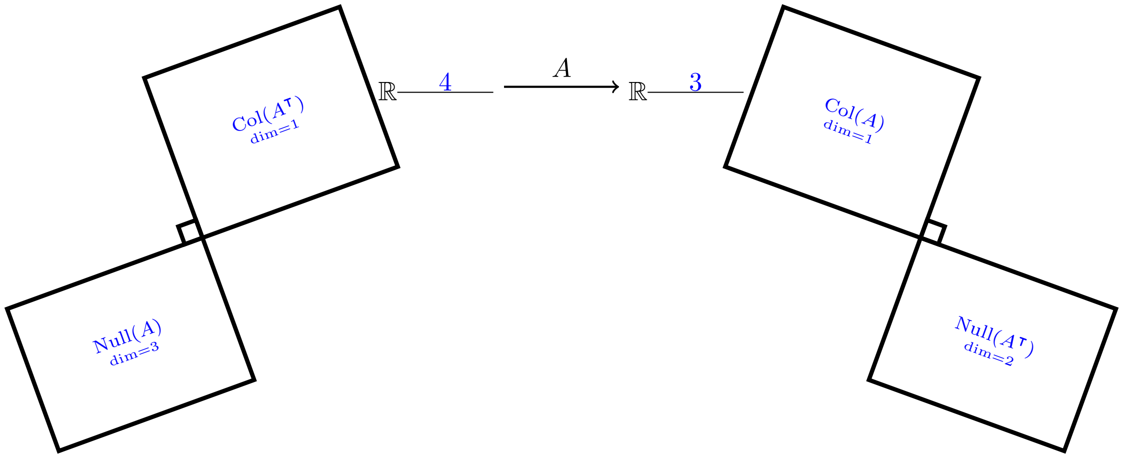
(7 pts) (*a*) If possible, determine if $\begin{bmatrix}2 & 4 & 3\end{bmatrix}^\intercal \in \operatorname{Null}(A^\intercal)$. If this is not possible, then explain why.

Solution. Vectors in $\operatorname{Null}(A^\intercal)$ must be orthogonal to the column space $\operatorname{Col}(A)$. We are told that $\begin{bmatrix}1 & 3 & -1\end{bmatrix}^\intercal$ is the first column, so the inner product

$$\langle \begin{bmatrix}2 & 4 & 3\end{bmatrix}^\intercal, \begin{bmatrix}1 & 3 & -1\end{bmatrix}^\intercal \rangle = 11 \neq 0$$

tells us that $\begin{bmatrix}2 & 4 & 3\end{bmatrix}^\intercal \notin \operatorname{Null}(A^\intercal)$.

(10 pts) (*b*) Now, suppose that $\operatorname{Null}(A) = \operatorname{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ where $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent. Fill in every missing label in the picture of the four fundamental subspaces below, including the dimension of each fundamental subspace.



The fact that $\begin{bmatrix}1 & -1 & 0 & 1\end{bmatrix}^\intercal$ is the first row and $\begin{bmatrix}1 & 3 & -1\end{bmatrix}^\intercal$ is the first column means that A looks like

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 3 & * & * & * \\ -1 & * & * & * \end{bmatrix}$$

So, A is 3×4, which means $\mathbb{R}^4 \xrightarrow{A} \mathbb{R}^3$. Furthermore, we are told that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis of $\operatorname{Null}(A)$, so $\dim \operatorname{Null}(A) = 3$. The rest of the dimensions follow from the rank-nullity theorem.

(8 pts) (*c*) Suppose again that $\operatorname{Null}(A) = \operatorname{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ where $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent. Find A.

Solution. The picture of the four fundamental subspaces tells us that $\operatorname{rank}(A) = \dim \operatorname{Col}(A) = 1$, which means every column of A is a multiple of the first column. Since we are given the first row of A, we have

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 3 & -3 & 0 & 3 \\ -1 & 1 & 0 & -1 \end{bmatrix}$$

Problem 4. A matrix A has projection onto $\operatorname{Col}(A^\intercal)$ and projection onto $\operatorname{Col}(A)$ given by

$$P_{\operatorname{Col}(A^\intercal)} = \begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 \\ -1/3 & 1/3 & 0 & 1/3 \\ -1/3 & 0 & 1/3 & -1/3 \\ 0 & 1/3 & -1/3 & 2/3 \end{bmatrix} \qquad P_{\operatorname{Col}(A)} = \begin{bmatrix} 5/9 & 4/9 & -2/9 \\ 4/9 & 5/9 & 2/9 \\ -2/9 & 2/9 & * \end{bmatrix}$$

Note that the $(3,3)$ entry of $P_{\operatorname{Col}(A)}$ is unknown.

(9 pts) (a) Is the system $A^\intercal \boldsymbol{x} = \begin{bmatrix} 3 & 3 & 0 & 0 \end{bmatrix}^\intercal$ consistent?

Solution. This is equivalent to asking if $\begin{bmatrix} 3 & 3 & 0 & 0 \end{bmatrix}^\intercal \in \operatorname{Col}(A^\intercal)$, which we can verify using the projection matrix $P_{\operatorname{Col}(A^\intercal)}$. Here, the relevant calculation is

$$\overbrace{\begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 \\ -1/3 & 1/3 & 0 & 1/3 \\ -1/3 & 0 & 1/3 & -1/3 \\ 0 & 1/3 & -1/3 & 2/3 \end{bmatrix}}^{P_{\operatorname{Col}(A^\intercal)}} \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Projecting $\begin{bmatrix} 3 & 3 & 0 & 0 \end{bmatrix}^\intercal$ onto $\operatorname{Col}(A^\intercal)$ changes $\begin{bmatrix} 3 & 3 & 0 & 0 \end{bmatrix}^\intercal$, so $\begin{bmatrix} 3 & 3 & 0 & 0 \end{bmatrix}^\intercal \notin \operatorname{Col}(A^\intercal)$.

(9 pts) (b) Find the projection of $\begin{bmatrix} 9 & 0 & 0 \end{bmatrix}^\intercal$ onto $\operatorname{Null}(A^\intercal)$.

Solution. The projection of $\begin{bmatrix} 9 & 0 & 0 \end{bmatrix}^\intercal$ onto $\operatorname{Col}(A)$ is

$$\overbrace{\begin{bmatrix} 5/9 & 4/9 & -2/9 \\ 4/9 & 5/9 & 2/9 \\ -2/9 & 2/9 & * \end{bmatrix}}^{P_{\operatorname{Col}(A)}} \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$$

Our orthogonality conditions then tell us that the projection of $\begin{bmatrix} 9 & 0 & 0 \end{bmatrix}^\intercal$ onto $\operatorname{Null}(A^\intercal)$ is

$$\begin{bmatrix} 9 & 0 & 0 \end{bmatrix}^\intercal - \begin{bmatrix} 5 & 4 & -2 \end{bmatrix}^\intercal = \begin{bmatrix} 4 & -4 & 2 \end{bmatrix}^\intercal$$

(7 pts) (c) Find the missing $(3,3)$ entry of $P_{\operatorname{Col}(A)}$. *Hint.* Start by explaining the relationship between the the dimensions of $\operatorname{Col}(A^\intercal)$ and $\operatorname{Col}(A)$. What property of projection matrices relates to dimension?

Solution. We know the general formula $\operatorname{trace}(P_V) = \dim(V)$. Since $\dim \operatorname{Col}(A^\intercal) = \dim \operatorname{Col}(A) = \operatorname{rank}(A)$, our two given projection matrices must have the same trace, which gives

$$\frac{2+1+1+2}{3} = \frac{5+5+9*}{9}$$

This reduces to $2 = {}^{(10+9*)}/_9$ which gives $* = 8/9$.

Problem 5. The following QR-factorization was calculated using the Gram-Schmidt algorithm.

$$\overset{A}{\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 5 \\ 1 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix}} = \overset{Q}{\begin{bmatrix} 0 & * & 1/2 \\ 1/\sqrt{2} & * & 1/2 \\ 1/\sqrt{2} & * & -1/2 \\ 0 & * & 1/2 \end{bmatrix}} \overset{R}{\begin{bmatrix} \sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{bmatrix}}$$

Note that the second column of Q is missing.

(9 pts) (*a*) $\operatorname{rank}(A) =$ __3__, $\operatorname{rank}(Q) =$ __3__, and $\operatorname{rank}(R) =$ __3__

(8 pts) (*b*) Use the Gram-Schmidt algorithm to find the missing column of Q. You must use the Gram-Schmidt algorithm to receive any credit.

Solution. We apply Gram-Schmidt to the columns of A. This gives

$$\begin{aligned} \boldsymbol{w}_1 &= \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^\intercal \\ \boldsymbol{w}_2 &= \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}^\intercal - \operatorname{proj}_{\boldsymbol{w}_1}(\begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}^\intercal) \\ &= \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}^\intercal - \frac{\langle \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^\intercal, \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}^\intercal \rangle}{\langle \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^\intercal, \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^\intercal \rangle} \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^\intercal \\ &= \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}^\intercal - \frac{2}{2} \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^\intercal \\ &= \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}^\intercal \end{aligned}$$

Normalizing $\boldsymbol{w}_2$ gives our missing column of Q as $\boldsymbol{q}_2 = \begin{bmatrix} 1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}^\intercal$.

(8 pts) (*c*) The vector $\boldsymbol{b} = \begin{bmatrix} 2 & 2 & 0 & 0 \end{bmatrix}^\intercal$ is orthogonal to the second column of Q. Find the least-squares approximate solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Solution. The least squares system $A^\intercal A\widehat{\boldsymbol{x}} = A^\intercal \boldsymbol{b}$ reduces to $R\widehat{\boldsymbol{x}} = Q^\intercal \boldsymbol{b}$. We are told that the missing column of Q is orthogonal to $\boldsymbol{b}$, so we can calculate

$$\overset{Q^\intercal}{\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ * & * & * & * \\ 1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix}} \overset{\boldsymbol{b}}{\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 2 \end{bmatrix}$$

The system $R\widehat{\boldsymbol{x}} = Q^\intercal \boldsymbol{b}$ is then solved with back-substitution.

$$\begin{array}{rcrcrcrcl} \sqrt{2}\,\widehat{x}_1 & + & \sqrt{2}\,\widehat{x}_2 & + & 2\sqrt{2}\,\widehat{x}_3 & = & \sqrt{2} & \to & \widehat{x}_1 = -3 \\ & & 2\,\widehat{x}_2 & - & 4\,\widehat{x}_3 & = & 0 & \to & \widehat{x}_2 = 2 \\ & & & & 2\,\widehat{x}_3 & = & 2 & \to & \widehat{x}_3 = 1 \end{array}$$

This gives $\widehat{x} = \begin{bmatrix} -3 & 2 & 1 \end{bmatrix}^\intercal$.