Duke University

Math 218

Matrices and Vector Spaces

Exam II

Name: ______________________ *NetID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

October 22, 2021

- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$.

(6 pts) (*a*) Determine if $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\intercal \in \text{Null}(A)$.

(6 pts) (*b*) Determine if $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\intercal \in \text{Col}(A)$.

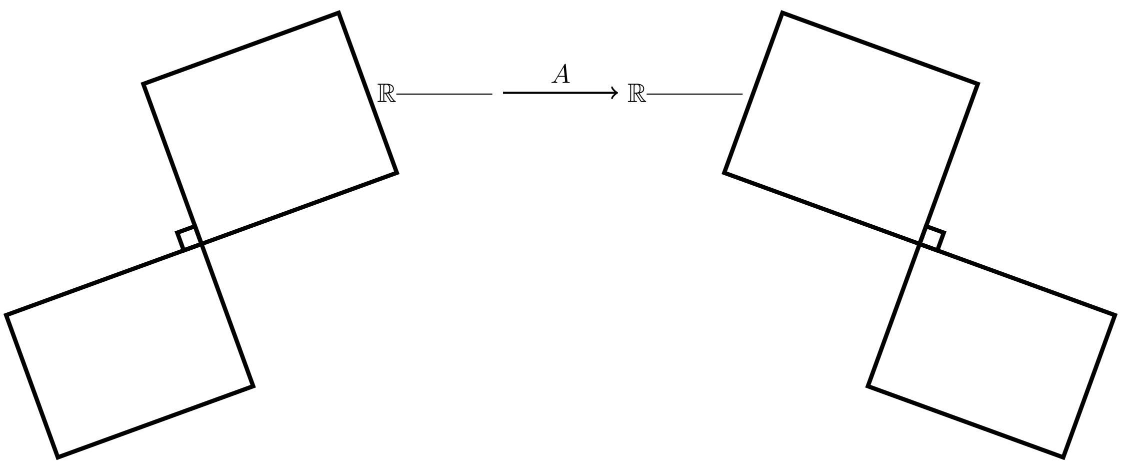
(6 pts) (*c*) Is $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\intercal$ an eigenvector of A? If so, what is the associated eigenvalue?

(7 pts) **Problem 2.** Suppose that A is a matrix and $\boldsymbol{v}$ is a vector satisfying $\boldsymbol{v} \in \mathcal{E}_A(-3)$. Show that $\boldsymbol{v}$ is an eigenvector of $M = A^2 - A$ and identify the corresponding eigenvalue.

Problem 3. The first row of a matrix A is the vector $\begin{bmatrix}1 & -1 & 0 & 1\end{bmatrix}^\intercal$ and the first column is the vector $\begin{bmatrix}1 & 3 & -1\end{bmatrix}^\intercal$.

(7 pts) (a) If possible, determine if $\begin{bmatrix}2 & 4 & 3\end{bmatrix}^\intercal \in \operatorname{Null}(A^\intercal)$. If this is not possible, then explain why.

(10 pts) (b) Now, suppose that $\operatorname{Null}(A) = \operatorname{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ where $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent. Fill in every missing label in the picture of the four fundamental subspaces below, including the dimension of each fundamental subspace.



(8 pts) (c) Suppose again that $\operatorname{Null}(A) = \operatorname{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ where $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent. Find A.

Problem 4. A matrix A has projection onto $\mathrm{Col}(A^\intercal)$ and projection onto $\mathrm{Col}(A)$ given by

$$P_{\mathrm{Col}(A^\intercal)} = \begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 \\ -1/3 & 1/3 & 0 & 1/3 \\ -1/3 & 0 & 1/3 & -1/3 \\ 0 & 1/3 & -1/3 & 2/3 \end{bmatrix} \qquad P_{\mathrm{Col}(A)} = \begin{bmatrix} 5/9 & 4/9 & -2/9 \\ 4/9 & 5/9 & 2/9 \\ -2/9 & 2/9 & * \end{bmatrix}$$

Note that the $(3,3)$ entry of $P_{\mathrm{Col}(A)}$ is unknown.

(9 pts) (a) Is the system $A^\intercal \boldsymbol{x} = \begin{bmatrix} 3 & 3 & 0 & 0 \end{bmatrix}^\intercal$ consistent?

(9 pts) (b) Find the projection of $\begin{bmatrix} 9 & 0 & 0 \end{bmatrix}^\intercal$ onto $\mathrm{Null}(A^\intercal)$.

(7 pts) (c) Find the missing $(3,3)$ entry of $P_{\mathrm{Col}(A)}$. *Hint.* Start by explaining the relationship between the the dimensions of $\mathrm{Col}(A^\intercal)$ and $\mathrm{Col}(A)$. What property of projection matrices relates to dimension?

Problem 5. The following QR-factorization was calculated using the Gram-Schmidt algorithm.

$$\overset{A}{\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 5 \\ 1 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix}} = \overset{Q}{\begin{bmatrix} 0 & * & 1/2 \\ 1/\sqrt{2} & * & 1/2 \\ 1/\sqrt{2} & * & -1/2 \\ 0 & * & 1/2 \end{bmatrix}} \overset{R}{\begin{bmatrix} \sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{bmatrix}}$$

Note that the second column of Q is missing.

(9 pts) (*a*) $\operatorname{rank}(A) =$ _______, $\operatorname{rank}(Q) =$ _______, and $\operatorname{rank}(R) =$ _______

(8 pts) (*b*) Use the Gram-Schmidt algorithm to find the missing column of Q. You must use the Gram-Schmidt algorithm to receive any credit.

(8 pts) (*c*) The vector $\boldsymbol{b} = \begin{bmatrix} 2 & 2 & 0 & 0 \end{bmatrix}^\intercal$ is orthogonal to the second column of Q. Find the least-squares approximate solution to $A\boldsymbol{x} = \boldsymbol{b}$.