Duke University

Math 218

Matrices and Vector Spaces

Exam III

Name: Solutions

NetID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

November 19, 2021

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the invertible matrix A and its cofactor matrix C given by

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 & -2 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ -2 & -1 & -2 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 & 1 & 2 & 3 \\ -3 & 0 & 3 & -3 & 0 \\ 3 & * & * & 3 & 3 \\ 2 & 1 & * & * & 0 \\ 0 & -3 & 3 & 3 & 3 \end{bmatrix}$$

Note that C is missing several entries.

(10 pts) (a) Find the missing $(4,3)$ entry of C.

Solution. This is the $(4,3)$ cofactor of A, which is $C_{43} = (-1)^{4+3} \cdot |A_{43}| = -1 \cdot |A_{43}|$. The missing entry of C is then

$$* = -1 \cdot \begin{vmatrix} 1 & -1 & 0 & -2 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{vmatrix} \xrightarrow[\boldsymbol{r}_3 + \boldsymbol{r}_1 \to \boldsymbol{r}_3]{\boldsymbol{r}_2 + \boldsymbol{r}_1 \to \boldsymbol{r}_2} -1 \cdot \begin{vmatrix} 1 & -1 & 0 & -2 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 1 \end{vmatrix} \xrightarrow{\boldsymbol{r}_4 + \boldsymbol{r}_3 \to \boldsymbol{r}_4} -1 \cdot \begin{vmatrix} 1 & -1 & 0 & -2 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -1$$

(10 pts) (b) Find $\chi_A(0)$ (the constant coefficient of the characteristic polynomial of A). *Hint.* This can be done by calculating a single inner product.

Solution. We know from class that $\chi_A(0) = (-1)^5 \cdot \det(A) = -\det(A)$. We also know that $AC^\intercal = A\,\mathrm{adj}(A) = \det(A) \cdot I_5$, which means that $\det(A)$ can be calculated as the inner product of the ith row of A and the ith row of C. So, the constant coefficient of the characteristic polynomial of A is

$$\chi_A(0) = -\det(A) = -\langle \begin{bmatrix} 1 & -1 & 1 & 0 & -2 \end{bmatrix}^\intercal, \begin{bmatrix} 1 & -1 & 1 & 2 & 3 \end{bmatrix}^\intercal \rangle = 3$$

(10 pts) (c) Find the solution $\boldsymbol{x}$ to the system $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = \det(A) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \end{bmatrix}^\intercal$.

Solution. Our matrix A is invertible, so we have a single solution given by

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \frac{1}{\det(A)} C^\intercal \det(A) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = C^\intercal \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

(20 pts) **Problem 2.** Consider the system of differential equations given by

$$\begin{aligned} f' &= 5f - 3g \qquad & f(0) &= 1 \\ g' &= 6f - 4g \qquad & g(0) &= -2 \end{aligned}$$

Find $f(t)$ and $g(t)$.

Solution. This initial value problem is $\boldsymbol{u}' = A\boldsymbol{u}$ with $\boldsymbol{u}(0) = \boldsymbol{u}_0$ where

$$\boldsymbol{u} = \begin{bmatrix} f(t) \\ g(g) \end{bmatrix} \qquad A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} \qquad \boldsymbol{u}_0 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The solution is $\boldsymbol{u}(t) = \exp(At)\boldsymbol{u}_0$. To calculate this matrix exponential, we must diagonalize A. We start with the characteristic polynomial

$$\chi_A(t) = t^2 - \operatorname{trace}(A)\,t + \det(A) = t^2 - t - 2 = (t-2)\cdot(t+1)$$

So, we have E-Vals$(A) = \{2, -1\}$. The eigenspaces are

$$\mathcal{E}_A(2) = \operatorname{Null}\overset{2\cdot I_2 - A}{\begin{bmatrix} -3 & 3 \\ -6 & 6 \end{bmatrix}} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \qquad \mathcal{E}_A(-1) = \operatorname{Null}\overset{-1\cdot I_2 - A}{\begin{bmatrix} -6 & 3 \\ -6 & 3 \end{bmatrix}} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

This gives our diagonalization

$$\overset{A}{\begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}} = \overset{X}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}} \overset{D}{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}$$

Now, the solution to our initial value problem is

$$\begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = \overset{X}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}} \overset{\exp(Dt)}{\begin{bmatrix} e^{2t} & \\ & e^{-t} \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}} \overset{\boldsymbol{u}_0}{\begin{bmatrix} 1 \\ -2 \end{bmatrix}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{2t} & \\ & e^{-t} \end{bmatrix} \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4e^{2t} \\ -3e^{-t} \end{bmatrix} = \begin{bmatrix} 4e^{2t} - 3e^{-t} \\ 4e^{2t} - 6e^{-t} \end{bmatrix}$$

Problem 3. The matrix H below is Hermitian with exactly two eigenvalues $\text{E-Vals}(H) = \{\lambda_1, \lambda_2\}$ where $\lambda_1 = 2$ and λ_2 is unknown. A basis of $\mathcal{E}_H(2)$ is given below.

$$H = \begin{bmatrix} 8 & * & -2 & * \\ * & 8 & * & * \\ -2 & * & 8 & * \\ * & * & -2i & 8 \end{bmatrix} \qquad \mathcal{E}_H(2) = \operatorname{Span}\{\begin{bmatrix} i & -i & i & -1 \end{bmatrix}^\intercal\}$$

Note that H is missing several entries.

(3 pts) (a) The $(3, 4)$ entry of H is $\underline{\overline{-2i} = 2i}$.

(3 pts) (b) The algebraic multiplicity of $\lambda_1 = 2$ as an eigenvalue of H is $\underline{\quad 1 \quad}$.

(3 pts) (c) The algebraic multiplicity of λ_2 as an eigenvalue of H is $\underline{\quad 3 \quad}$.

(3 pts) (d) The coefficient of t^3 in $\chi_H(t)$ is $\underline{-\operatorname{trace}(H) = -32}$.

(8 pts) (e) Determine if $\begin{bmatrix} i & 1 & -i & -i \end{bmatrix}^\intercal \in \mathcal{E}_H(\lambda_2)$. Explain your resoning.

Solution. The Spectral Theorem says that $\mathcal{E}_H(\lambda_2)$ is orthogonal to $\mathcal{E}_H(2)$. The inner product

$$\begin{aligned} \langle \begin{bmatrix} i & 1 & -i & -i \end{bmatrix}^\intercal, \begin{bmatrix} i & -i & i & -1 \end{bmatrix}^\intercal \rangle &= \overline{(i)} \cdot (i) + \overline{(1)} \cdot (-i) + \overline{(-i)} \cdot (i) + \overline{(-i)} \cdot (-1) \\ &= (1) + (-i) + (-1) + (-i) \\ &= -2i \\ &\neq 0 \end{aligned}$$

then tells us that $\begin{bmatrix} i & 1 & -i & -i \end{bmatrix}^\intercal \notin \mathcal{E}_H(\lambda_2)$.

(8 pts) (f) Find λ_2. Explain your reasoning.

Solution. We know that $\operatorname{trace}(H) = \operatorname{am}_H(\lambda_1) \cdot \lambda_1 + \operatorname{am}_H(\lambda_2) \cdot \lambda_2$, which gives

$$32 = 1 \cdot 2 + 3 \cdot \lambda_2$$

This implies that $\lambda_2 = 10$.

Problem 4. Consider the quadratic form on $\mathbb{R}^3$ given by

$$q(x_1, x_2, x_3) = (x_1 + 4\,x_2 + 5\,x_3)^2 + (3\,x_1 - 2\,x_2 + x_3)^2 + (-2\,x_1 + x_2 - x_3)^2 + (2\,x_1 + 5\,x_2 + 7\,x_3)^2$$

Note that this quadratic form may be written as $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$ where S is real-symmetric.

(3 pts) (a) Which of the following adjectives correctly describes $q(\boldsymbol{x})$?

√ positive semidefinite ◯ negative semidefinite ◯ indefinite

(3 pts) (b) Which of the following correctly describes the eigenvalues of S?

◯ Some eigenvalues of S are positive and some eigenvalues of S are negative.

◯ The eigenvalues of S are all nonpositive.

√ The eigenvalues of S are all nonnegative.

(10 pts) (c) If possible, find A such that $S = A^\intercal A$. If this is not possible, then explain why.

Solution. Recall that $S = A^\intercal A$ implies $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle = \langle \boldsymbol{x}, A^\intercal A\boldsymbol{x} \rangle = \langle A\boldsymbol{x}, A\boldsymbol{x} \rangle = \|A\boldsymbol{x}\|^2$. Our quadratic form is given to us as a sum of squares, so we have $S = A^\intercal A$ where

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & -2 & 1 \\ -2 & 1 & -1 \\ 2 & 5 & 7 \end{bmatrix}$$

(6 pts) (d) If possible, find $\boldsymbol{x} \neq \boldsymbol{O}$ satisfying $q(\boldsymbol{x}) = 0$. If this is not possible, then explain why.

Solution. From (c), we know that $q(\boldsymbol{x}) = \|A\boldsymbol{x}\|^2$. This means that any $\boldsymbol{x} \neq \boldsymbol{O}$ in $\operatorname{Null}(A)$ will do. Noting that the third column of A is the sum of the first two then tells us that $q(1, 1, -1) = 0$.