Duke University

Math 218

Matrices and Vector Spaces

Exam III

Name: ______________________ *NetID:* ______________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

November 19, 2021

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the invertible matrix A and its cofactor matrix C given by

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 & -2 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ -2 & -1 & -2 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 & 1 & 2 & 3 \\ -3 & 0 & 3 & -3 & 0 \\ 3 & * & * & 3 & 3 \\ 2 & 1 & * & * & 0 \\ 0 & -3 & 3 & 3 & 3 \end{bmatrix}$$

Note that C is missing several entries.

(10 pts) (*a*) Find the missing $(4, 3)$ entry of C.

(10 pts) (*b*) Find $\chi_A(0)$ (the constant coefficient of the characteristic polynomial of A). *Hint.* This can be done by calculating a single inner product.

(10 pts) (*c*) Find the solution $\boldsymbol{x}$ to the system $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = \det(A) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \end{bmatrix}^{\intercal}$.

(20 pts) **Problem 2.** Consider the system of differential equations given by

$$\begin{aligned} f' &= 5\,f - 3\,g \qquad & f(0) &= 1 \\ g' &= 6\,f - 4\,g \qquad & g(0) &= -2 \end{aligned}$$

Find $f(t)$ and $g(t)$.

Problem 3. The matrix H below is Hermitian with exactly two eigenvalues E-Vals$(H) = \{\lambda_1, \lambda_2\}$ where $\lambda_1 = 2$ and λ_2 is unknown. A basis of $\mathcal{E}_H(2)$ is given below.

$$H = \begin{bmatrix} 8 & * & -2 & * \\ * & 8 & * & * \\ -2 & * & 8 & * \\ * & * & -2i & 8 \end{bmatrix} \qquad \mathcal{E}_H(2) = \operatorname{Span}\{\begin{bmatrix} i & -i & i & -1 \end{bmatrix}^\intercal\}$$

Note that H is missing several entries.

(3 pts) (a) The $(3, 4)$ entry of H is ________________.

(3 pts) (b) The algebraic multiplicity of $\lambda_1 = 2$ as an eigenvalue of H is ________.

(3 pts) (c) The algebraic multiplicity of λ_2 as an eigenvalue of H is ________.

(3 pts) (d) The coefficient of t^3 in $\chi_H(t)$ is ________________.

(8 pts) (e) Determine if $\begin{bmatrix} i & 1 & -i & -i \end{bmatrix}^\intercal \in \mathcal{E}_H(\lambda_2)$. Explain your resoning.

(8 pts) (f) Find λ_2. Explain your reasoning.

Problem 4. Consider the quadratic form on $\mathbb{R}^3$ given by

$$q(x_1, x_2, x_3) = (x_1 + 4\,x_2 + 5\,x_3)^2 + (3\,x_1 - 2\,x_2 + x_3)^2 + (-2\,x_1 + x_2 - x_3)^2 + (2\,x_1 + 5\,x_2 + 7\,x_3)^2$$

Note that this quadratic form may be written as $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$ where S is real-symmetric.

(3 pts) (a) Which of the following adjectives correctly describes $q(\boldsymbol{x})$?

○ positive semidefinite ○ negative semidefinite ○ indefinite

(3 pts) (b) Which of the following correctly describes the eigenvalues of S?

○ Some eigenvalues of S are positive and some eigenvalues of S are negative.
○ The eigenvalues of S are all nonpositive.
○ The eigenvalues of S are all nonnegative.

(10 pts) (c) If possible, find A such that $S = A^\intercal A$. If this is not possible, then explain why.

(6 pts) (d) If possible, find $\boldsymbol{x} \neq \boldsymbol{O}$ satisfying $q(\boldsymbol{x}) = 0$. If this is not possible, then explain why.