Math 218D: Week 1 Discussion

Study Copy

August 26, 2021

Problem 1. Fill in the blanks below.

$$A = \begin{bmatrix} 3 & -4 & 2 & 8 & 0 \\ 1 & 8 & 1/9 & 0 & 1 \\ 2 & 4 & -1 & \pi & 2 \end{bmatrix} \qquad a_{23} = \underline{\hspace{1cm}} \qquad \mathrm{Col}_2(A) \in \mathbb{R}^{\underline{\hspace{0.7cm}}} \qquad A^{\intercal} = \underline{\hspace{6cm}}$$

Problem 2. Fill in the blanks below, assuming that S is *symmetric*.

$$S = \begin{bmatrix} 5 & -4 & \underline{\hspace{0.7cm}} & \underline{\hspace{0.7cm}} \\ \underline{\hspace{0.7cm}} & 19 & \underline{\hspace{0.7cm}} & -1 \\ 11 & 2 & 8 & 3 \\ 9 & \underline{\hspace{0.7cm}} & \underline{\hspace{0.7cm}} & -10 \end{bmatrix} \qquad \mathrm{trace}(S) = \underline{\hspace{0.7cm}}$$

Problem 3. By definition, a matrix S is *symmetric* if $\underline{\hspace{3cm}}$.

Problem 4. Suppose that A is $n \times n$ and let $S = A + A^{\intercal}$. Prove that S is symmetric.

Hint. This proof can be quickly accomplished by filling in the blanks below.

$$\underline{\hspace{3cm}} = \underline{\hspace{3cm}} = \underline{\hspace{3cm}} = \underline{\hspace{3cm}} = \underline{\hspace{3cm}}$$

Problem 5. Consider the matrix R given by

$$R = \begin{bmatrix} 1 & -3 & 0 & -9 & 5 \\ 0 & 0 & 1 & 14 & 9 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{Col}_4(R) = \underline{\hspace{0.7cm}}\ \mathrm{Col}_1(R) + \underline{\hspace{0.7cm}}\ \mathrm{Col}_3(R)$$

(a) Fill in the blanks above to express the fourth column of R as a linear combination of the first and third columns of R.

(b) Can the fifth column of R be expressed as a linear combination of the first and third columns of R? Explain why or why not.

Problem 6. We write $\mathbb{R}^9 \xrightarrow{A} \mathbb{R}^{22}$ to indicate that A is a $\underline{\quad\;}\times\underline{\quad\;}$ matrix.

Problem 7. Suppose $\mathbb{R}^{13} \xrightarrow{M^\intercal} \mathbb{R}^{37}$. Then M is a $\underline{\quad\;}\times\underline{\quad\;}$ matrix.

Problem 8. Fill in the blanks in the two equations below.

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\begin{bmatrix}7\\ 8\\ 9\end{bmatrix} = \underline{\quad\;}\begin{bmatrix}\underline{\quad\;}\\ \underline{\quad\;}\end{bmatrix} + \underline{\quad\;}\begin{bmatrix}\underline{\quad\;}\\ \underline{\quad\;}\end{bmatrix} + \underline{\quad\;}\begin{bmatrix}\underline{\quad\;}\\ \underline{\quad\;}\end{bmatrix} \qquad \begin{bmatrix}\underline{\quad\;} & \underline{\quad\;}\\ \underline{\quad\;} & \underline{\quad\;}\\ \underline{\quad\;} & \underline{\quad\;}\end{bmatrix}\begin{bmatrix}\underline{\quad\;}\\ \underline{\quad\;}\end{bmatrix} = 11\begin{bmatrix}-3\\ 5\\ 1\end{bmatrix} - 42\begin{bmatrix}7\\ 6\\ 4\end{bmatrix}$$

Problem 9. Fill in the blanks in each equation below.

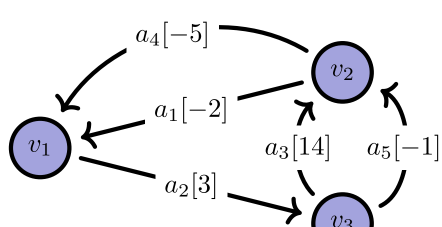
$$\begin{bmatrix}\; A \;\end{bmatrix}\begin{bmatrix}\underline{\quad\;}\\ \underline{\quad\;}\\ \underline{\quad\;}\end{bmatrix} = \text{the third column of } A \qquad \begin{bmatrix}\; A \;\end{bmatrix}\begin{bmatrix}\underline{\quad\;}\\ \underline{\quad\;}\\ \underline{\quad\;}\end{bmatrix} = \text{the first column minus the third column of } A$$

$$\begin{bmatrix}\; A \;\end{bmatrix}\begin{bmatrix}\underline{\quad\;}\\ \underline{\quad\;}\\ \underline{\quad\;}\end{bmatrix} = \text{the sum of all columns of } A \qquad \begin{bmatrix}\; A \;\end{bmatrix}\begin{bmatrix}\underline{\quad\;}\\ \underline{\quad\;}\\ \underline{\quad\;}\end{bmatrix} = \text{twice the first column of } A$$

Problem 10. Suppose that A has four columns. Fill in the blanks in the equation on the right to validate the equation on the left.

$$\operatorname{Col}_1 + 3\operatorname{Col}_2 - 9\operatorname{Col}_3 = 6\operatorname{Col}_4 \qquad \begin{bmatrix}\; A \;\end{bmatrix}\begin{bmatrix}\underline{\quad\;}\\ \underline{\quad\;}\\ \underline{\quad\;}\\ \underline{\quad\;}\end{bmatrix} = \boldsymbol{O}$$

Problem 11. Consider the weighted digraph G depicted below.



Use a matrix-vector product to calculate the weighted net flow through each node of G.

Math 218D: Week 2 Discussion

Study Copy

September 2, 2021

Problem 1. $\langle [1 \quad -3 \quad 0 \quad 2]^\intercal, [2 \quad 1 \quad 5 \quad 0]^\intercal \rangle =$ ____

Problem 2. Which of the following vectors is *orthogonal* to $\boldsymbol{v} = [1 \quad 1 \quad 1 \quad 1 \quad 1]^\intercal$?

○ $\boldsymbol{w} = [3 \quad -5 \quad 2 \quad 1 \quad 0]^\intercal$ ○ $\boldsymbol{x} = [9 \quad 2 \quad 3 \quad -6 \quad -8]^\intercal$ ○ $\boldsymbol{y} = [-1 \quad -1 \quad 9 \quad -10 \quad 3]^\intercal$

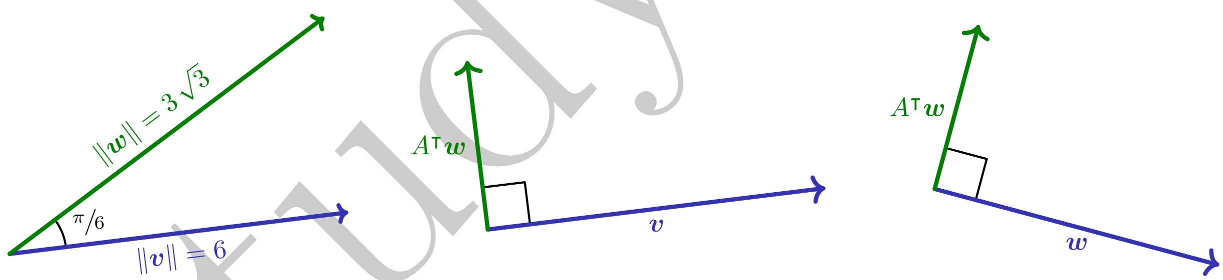
Problem 3. The length of $\boldsymbol{v}$ can be calculated with an inner product using the formula $\|\boldsymbol{v}\| =$ ______________.

Problem 4. The inner product can be interpreted geometrically with the formula $\langle \boldsymbol{v}, \boldsymbol{w} \rangle =$ ____________________.

Problem 5. If we view $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ as $n \times 1$ matrices, then $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ can be calculated using matrix multiplication with the formula $\langle \boldsymbol{v}, \boldsymbol{w} \rangle =$ ____________________.

Problem 6. The adjoint formula for inner products states that $\langle A\boldsymbol{v}, \boldsymbol{w} \rangle =$ ____________________.

Problem 7. Suppose that A and B are matrices satisfying $A^\intercal B = I_n$ and that $\boldsymbol{v}$ and $\boldsymbol{w}$ vectors making the following diagrams accurate.



Calculate $\langle B\boldsymbol{v} - 3\,\boldsymbol{w}, 2\,A\boldsymbol{v} - A\boldsymbol{w} \rangle$.

Problem 8. One of the following calculations is possible and the other is not. Carry out the possible calculation.

$$\begin{bmatrix} 5 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 1 \\ 3 & -3 \end{bmatrix} = \underline{\hspace{5cm}} \qquad \begin{bmatrix} 5 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 3 \\ 1 & 1 & -3 \end{bmatrix} = \underline{\hspace{5cm}}$$

Problem 9. Fill in the blanks in each of the following two equations.

$$\begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix} \begin{bmatrix} 0 & -3 & 1 & 0 \\ 1 & 9 & 0 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 50 & 5 & -4 \\ 12 & 193 & -3 & 19 \end{bmatrix} \qquad \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 2 & -5 \\ 0 & 9 \end{bmatrix}$$

Problem 10. Suppose that A and B are 2021×2021. Prove that $S = B^{\intercal} A + A^{\intercal} B$ is symmetric.

Problem 11. The last column of a matrix A is $\begin{bmatrix} 0 & 3 & 4 \end{bmatrix}^{\intercal}$ and the Gramian of A is

$$G = \begin{bmatrix} 9 & \_\_\_ & -6 & \_\_\_ \\ 3 & 14 & 13 & \_\_\_ \\ \_\_\_ & \_\_\_ & 29 & \_\_\_ \\ 5 & 13 & 0 & \_\_\_ \end{bmatrix} = \underline{\hspace{3cm}}$$

(a) Fill in the missing entries of G and fill in the formula used to calculate G.

(b) The number of rows of A is ____ and the number of columns of A is ____.

(c) Which (if any) of the columns of A is orthogonal to the third column of A?

Math 218D: Week 3 Discussion

Study Copy

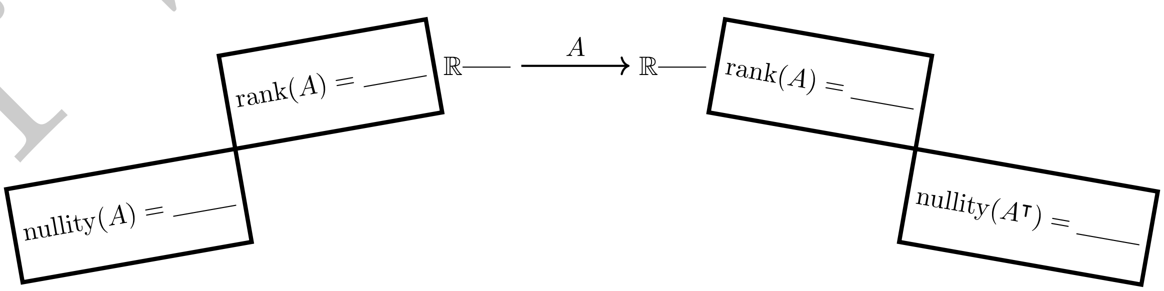
September 9, 2021

Problem 1. Consider the system of equations given by

$$\begin{array}{rcrcrcrcr} x_1 & + & 2\,x_2 & - & 4\,x_3 & + & 9\,x_4 & = & -2 \\ 5\,x_1 & + & 11\,x_2 & - & 13\,x_3 & + & 37\,x_4 & = & 7 \\ -\,3\,x_1 & - & 6\,x_2 & + & 12\,x_3 & - & 24\,x_4 & = & 0 \end{array}$$

Use the Gauß-Jordan algorithm to find the general solution to this system.

Problem 2. Suppose A is a matrix satisfying $\operatorname{rref}(A) = \begin{bmatrix} 1 & -13 & 0 & 6 \\ 0 & 0 & 1 & 17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Fill-in the blanks below.



Problem 3. Use the Gauß-Jordan algorithm to calculate $\operatorname{rref}(A)$ where $A = \begin{bmatrix} 3 & -6 & 12 & 0 & -9 \\ -7 & 14 & -28 & -5 & 26 \\ 5 & -12 & 12 & 2 & -13 \\ 2 & -3 & 12 & -3 & -5 \end{bmatrix}$.

Math 218D: Week 4 Discussion

Study Copy

September 16, 2021

Problem 1. Use the Gauß-Jordan algorithm to calculate $EA = R$ where $A = \begin{bmatrix} 1 & -5 & 0 & 3 \\ 0 & 0 & 5 & -5 \\ -5 & 25 & -11 & -4 \end{bmatrix}$.

Problem 2. Consider the $EA = R$ factorization and the vector $\boldsymbol{b}$ given by

$$\overset{E}{\begin{bmatrix} -6 & 5 & 2 & -13 \\ 4 & -4 & -2 & 9 \\ -9 & 9 & 4 & -21 \\ 1 & -2 & -1 & 3 \end{bmatrix}} \overset{A}{\begin{bmatrix} -3 & 21 & 2 & -26 & -5 & 3 \\ 3 & -21 & 1 & 14 & 38 & -2 \\ -3 & 21 & -3 & -6 & -60 & 1 \\ 2 & -14 & -1 & 16 & 7 & -2 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & -7 & 0 & 6 & 9 & 0 \\ 0 & 0 & 1 & -4 & 11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \qquad \boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Determine if $A\boldsymbol{x} = \boldsymbol{b}$ is consistent *without doing any row operations.*

Problem 3. Calculate $PA = LU$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & -2 \\ 3 & -1 & 1 \end{bmatrix}$.

Problem 4. Consider the $PA = LU$ factorization and the vector $\boldsymbol{b}$ given by

$$\overset{P}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}} \qquad \boldsymbol{b} = \begin{bmatrix} 15 \\ 11 \\ -15 \\ 13 \end{bmatrix}$$

Solve $A\boldsymbol{x} = \boldsymbol{b}$ *without doing any row reductions.*

Math 218D: Week 5 Discussion

Study Copy

September 23, 2021

Problem 1. To verify if $\boldsymbol{v} \in \mathrm{Null}(A)$ we must check the equation ____________________.

Problem 2. Show that $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is in the null space of $A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 3 \\ 2 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$.

Problem 3. Suppose A is 5×9 and that $\boldsymbol{v} \in \mathrm{Null}(A)$. Then $\boldsymbol{v} \in \mathbb{R}^{\_\_\_}$.

Problem 4. Suppose that A has four columns related by the equation $\mathrm{Col}_4 = 3\ \mathrm{Col}_1 + \mathrm{Col}_2 - \mathrm{Col}_3$. Find a nonzero vector $\boldsymbol{v} \in \mathrm{Null}(A)$.

Problem 5. A scalar λ is an *eigenvalue* of A if ____________________.

Problem 6. The *eigenspace* of A corresponding to an eigenvalue λ is $\mathcal{E}_A(\lambda) =$ ____________________.

Problem 7. An *eigenvector* $\boldsymbol{v} \in \mathcal{E}_A(\lambda)$ satisfies the equation ____________________.

Problem 8. Show that $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 10 & -6 & 6 \\ 6 & -2 & 6 \\ -6 & 6 & -2 \end{bmatrix}$ and identify the eigenvalue.

Problem 9. Find all vectors in $\mathcal{E}_A(-3)$ where $A = \begin{bmatrix} -23 & 40 & -60 \\ -5 & 7 & -15 \\ 5 & -10 & 12 \end{bmatrix}$.

Problem 10. To verify if $\boldsymbol{v} \in \operatorname{Col}(A)$ we must check the equation ________________________.

Problem 11. Determine if $\boldsymbol{v} = \begin{bmatrix} 6 \\ 12 \\ 2 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \\ -3 & 9 & -12 \end{bmatrix}$.

Problem 12. To verify if $\boldsymbol{v} \in \operatorname{Span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ we must check ____.

Problem 13. Determine if $\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}^\intercal \in \operatorname{Span}\{\begin{bmatrix} 1 & -3 & -3 \end{bmatrix}^\intercal, \begin{bmatrix} -1 & 4 & 3 \end{bmatrix}^\intercal\}$.

Math 218D: Week 6 Discussion

Study Copy

September 30, 2021

Problem 1. By definition, what does it mean to call a list of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ *linearly dependent*?

Problem 2. By definition, what does it mean to call a list of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ *linearly independent*?

Problem 3. Determine if $\{\begin{bmatrix}1 & -3 & 1\end{bmatrix}^\intercal, \begin{bmatrix}-4 & 13 & -3\end{bmatrix}^\intercal, \begin{bmatrix}5 & -17 & 3\end{bmatrix}^\intercal\}$ is independent.

Problem 4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbb{R}^n$ and let A be an $m \times n$ matrix such that $\{A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3\}$ is linearly independent. Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

Problem 5. The columns of a matrix A are independent if and only if ____________________________.

Problem 6. Consider the calculations

$$\operatorname{rref}\overbrace{\begin{bmatrix} 9 & 4 & 4 \\ -36 & -16 & -16 \\ 20 & 9 & 4 \\ -49 & -22 & -12 \end{bmatrix}}^{A} = \begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & -44 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \operatorname{rref}\overbrace{\begin{bmatrix} 9 & -36 & 20 & -49 \\ 4 & -16 & 9 & -22 \\ 4 & -16 & 4 & -12 \end{bmatrix}}^{A^\intercal} = \begin{bmatrix} 1 & -4 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find the pivot solutions to $A\boldsymbol{v} = \boldsymbol{O}$. These vectors form a basis of ______________.

(b) Find the pivot solutions to $A^\intercal\boldsymbol{v} = \boldsymbol{O}$. These vectors form a basis of ______________.

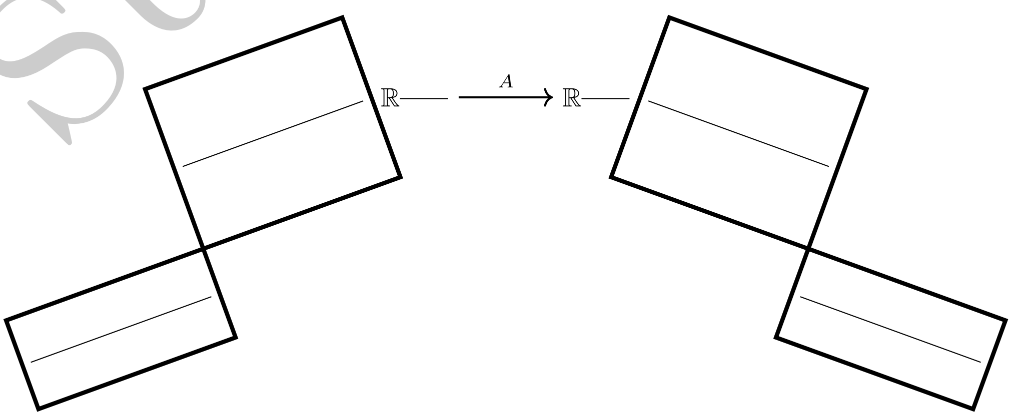
(c) Find the pivot columns of A. These vectors form a basis of ______________.

(d) Find the nonzero rows of $\operatorname{rref}(A)$. These vectors form a basis of ______________.

(e) The pivot columns of $A^\intercal$ form a basis of ______________.

(f) The nonzero rows of $\operatorname{rref}(A^\intercal)$ form a basis of ______________.

(g) Fill in the blanks in the figure below.



Math 218D: Week 7 Discussion

Study Copy

October 7, 2021

Problem 1. Suppose that A is a matrix satisfying

$$\mathrm{Col}(A^\intercal) = \mathrm{Span}\{\begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix}^\intercal, \begin{bmatrix}0 & 3 & 2 & 4\end{bmatrix}^\intercal\} \qquad \mathrm{Null}(A^\intercal) = \mathrm{Span}\{\begin{bmatrix}2 & 1 & 1\end{bmatrix}^\intercal\}$$

(a) Draw the picture of the four fundamental subspaces of A, including their dimensions

(b) Determine if $\boldsymbol{v} = \begin{bmatrix}1 & 0 & 2 & -1\end{bmatrix}^\intercal$ satisfies $A\boldsymbol{v} = \boldsymbol{O}$.

(c) Determine if $\boldsymbol{b} = \begin{bmatrix}3 & 5 & 2\end{bmatrix}^\intercal$ makes the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent.

(d) Explain why $\mathrm{Null}(A) \neq \mathrm{Span}\{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^\intercal, \begin{bmatrix}2 & 1 & 0 & 0\end{bmatrix}^\intercal\}$.

Problem 2. Suppose $EA = R$ where

$$E = \begin{bmatrix} 1 & -3 & -1 & 17 \\ -3 & 10 & 5 & -56 \\ 5 & -19 & -12 & 105 \\ -1 & 7 & 7 & -36 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -3 & 0 & -7 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Draw the picture of the four fundamental subspaces of A, including their dimensions

(b) Find a basis of $\mathrm{Col}(A^\intercal)$.

(c) Find a basis of $\mathrm{Null}(A^\intercal)$.

(d) Find a basis of $\mathrm{Null}(A)$.

(e) Find a basis of $\mathrm{Col}(A)$.

Math 218D: Week 8 Discussion

Study Copy

October 14, 2021

Problem 1. The *least squares problem* associated to $A\boldsymbol{x} = \boldsymbol{b}$ is ____________________________.

Problem 2. Suppose $\widehat{\boldsymbol{x}}$ is a least squares approximate solution to $A\boldsymbol{x} = \boldsymbol{b}$. Then $A\widehat{\boldsymbol{x}} =$ ________________________.

Problem 3. The *least squares error* is defined as ____________________________.

Problem 4. Define the concept of an $A = QR$ *factorization*.

Problem 5. A matrix M has orthonormal columns if and only if $M^{\intercal}M =$ ______________.

Problem 6. Given $A = QR$, projection onto $\text{Col}(A)$ is given by $P_{\text{Col}(A)} =$ ______________.

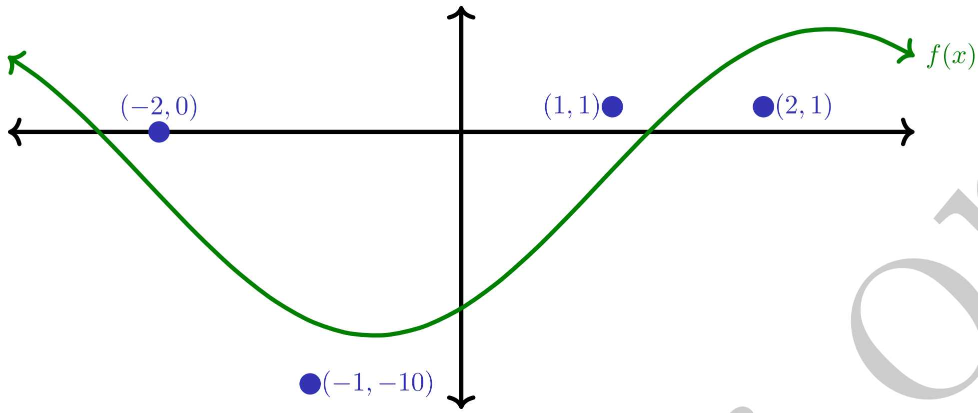
Problem 7. Suppose $A = QR$ where A has full column rank. Then the least squares problem $A^{\intercal}A\widehat{\boldsymbol{x}} = A^{\intercal}\boldsymbol{b}$ reduces to ____________________________.

Problem 8. Suppose that A is $m \times n$ with orthonormal columns and that $\boldsymbol{v} \in \mathbb{R}^n$.

(a) Show that $\|A\boldsymbol{v}\| = \|\boldsymbol{v}\|$.

(b) Show that $n \leq m$.

Problem 9. The figure below depicts the result of using the technique of least squares to fit a curve of the form $f(x) = c_0 + c_1\ \cos(\pi x/3) + c_2\ \sin(\pi x/3)$ to four data points.



Find the values of c_0, c_1, and c_2 and calculate the error in using $f(x)$ to approximate this data.

Math 218D: Week 9 Discussion

Study Copy

October 21, 2021

Problem 1. Calculate $\begin{vmatrix} 1 & -9 & 3 & -8 \\ 9 & -81 & 27 & -70 \\ -5 & 45 & -14 & 29 \\ -7 & 60 & -16 & 52 \end{vmatrix}$.

Problem 2. Consider the following matrix factorization

$$\overset{P}{\begin{bmatrix} 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 0&-1&-1&1&6 \\ -1&-4&1&-1&-4 \\ -2&-10&0&2&-1 \\ 8&3&1&-2&-3 \\ 3&-1&0&1&1 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1&0&0&0&0 \\ -1/4&1&0&0&0 \\ -1/8&29/74&1&0&0 \\ 0&4/37&-1&1&0 \\ 3/8&17/74&-8/19&-12/19&1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 8&3&1&-2&-3 \\ 0&-37/4&1/4&3/2&-7/4 \\ 0&0&38/37&-68/37&-273/74 \\ 0&0&0&-1&5/2 \\ 0&0&0&0&97/38 \end{bmatrix}}$$

Calculate $\det(A)$.

Problem 3. For $n \times n$ matrices A and B, $\det(A^\intercal) =$ ____________ and $\det(AB) =$ ____________.

Problem 4. If possible, find 3×3 matrices A and B satisfying $\det(A + B) \neq \det(A) + \det(B)$. If this is not possible, then explain why.

Problem 5. The (i, j) *minor* of A is $M_{ij} =$ ____________ and the (i, j) *cofactor* is $C_{ij} =$ ____________.

Problem 6. Suppose that $\det(A) = 35$ and that each (i,j) minor of A is the (i,j) entry of $M = \begin{bmatrix} -45 & -9 & -41 & 5 \\ -10 & 5 & 15 & 5 \\ 45 & 16 & 34 & -5 \\ -10 & -2 & -13 & 5 \end{bmatrix}$.

(a) Find the cofactor matrix C of A and the adjugate matrix $\operatorname{adj}(A)$.

(b) Find three independent vectors orthogonal to the first column of A.

(c) Solve $A\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{b} = \begin{bmatrix} 0 & 7 & 0 & 0 \end{bmatrix}^\intercal$.

Problem 7. Suppose that λ is an eigenvalue of an $n \times n$ matrix A. Then $\det(\lambda \cdot I_n - A) =$ ______________.

Math 218D: Week 10 Discussion

Study Copy

October 28, 2021

Problem 1. The reciprocal of $z = 7 - 9\,i$ is $1/z =$ ____________ + ____________ i.

Problem 2. Consider the vectors $\boldsymbol{v} = \begin{bmatrix} 1+i & 5 \end{bmatrix}^\intercal$ and $\boldsymbol{w} = \begin{bmatrix} 1-3\,i & 2+i \end{bmatrix}^\intercal$ and the matrix $A = \left[\begin{smallmatrix} 2 & 1+i & -1 \\ 0 & 1 & 3-2\,i \end{smallmatrix}\right]$.

(a) Calculate $\|\boldsymbol{v}\|$.

(b) Calculate $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$.

(c) Calculate $A^*\boldsymbol{v}$.

Problem 3. We call a matrix A *Hermitian* if ____________. We call A *unitary* if ____________.

Problem 4. Suppose that H is Hermitian. Show that every diagonal entry of H is a real number.

Problem 5. Suppose that U is $n \times n$ unitary and that $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}^n$. Show that $\langle U\boldsymbol{v}, U\boldsymbol{w}\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle$.

Problem 6. The polynomial

$$f(t) = t^4 - 2\,t^3 - t^2 + t - 14$$

has four distinct roots r_1, r_2, r_3, and r_4.

(a) $r_1 + r_2 + r_3 + r_4 =$ ____________________ and $r_1 r_2 r_3 r_4 =$ ____________________

(b) Calculate $(1 - r_1)(1 - r_2)(1 - r_3)(1 - r_4)$.

Problem 7. Let r_1 and r_2 be the roots of

$$f(t) = -9\,t^2 - 2\,t - 1$$

Calculate $r_1^2 + r_2^2$.

Hint. Consider $(r_1 + r_2)^2$.

Math 218D: Week 11 Discussion

Study Copy

November 4, 2021

Problem 1. Consider the equation

$$\overset{A}{\begin{bmatrix} 18 & 10 & -5 & 0 \\ -10 & * & 8 & 5 \\ -6 & -28 & -14 & -14 \\ -11 & -30 & -11 & -10 \end{bmatrix}} = \overset{X}{\begin{bmatrix} 18 & 10 & -5 & 0 \\ -10 & * & 8 & 5 \\ -6 & -28 & -14 & -14 \\ -11 & -30 & -11 & -10 \end{bmatrix}} \overset{B}{\begin{bmatrix} -1 & * & -7 & -1 \\ 0 & 0 & * & * \\ -1 & 6 & -1 & -6 \\ * & * & 2 & -1 \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} 11 & 10 & 0 & 3 \\ * & -7 & * & -3 \\ -10 & * & * & -3 \\ 3 & 3 & * & * \end{bmatrix}}$$

where the entries marked $*$ are unknown. Find the missing entry of A.

Problem 2. Suppose that A has eigenspaces given by

$$\mathcal{E}_A(7) = \operatorname{Span}\{\begin{bmatrix}1 & 3 & 0\end{bmatrix}^\intercal\} \quad \mathcal{E}_A(1) = \operatorname{Span}\{\begin{bmatrix}-2 & -5 & -5\end{bmatrix}^\intercal\} \quad \mathcal{E}_A(-1) = \operatorname{Span}\{\begin{bmatrix}-3 & -7 & -9\end{bmatrix}^\intercal\}$$

Calculate $A^{2021}\boldsymbol{v}$ for $\boldsymbol{v} = \begin{bmatrix}0 & -1 & 3\end{bmatrix}^\intercal$.

Problem 3. Consider the factorization

$$\overset{A}{\begin{bmatrix} -233 & 693 \\ -84 & 250 \end{bmatrix}} = \begin{bmatrix} 3 & 11 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 19 \end{bmatrix} \begin{bmatrix} 3 & 11 \\ 1 & 4 \end{bmatrix}^{-1}$$

(a) $\det(A) =$ ______________

(b) Find the solution $\boldsymbol{u}(t)$ to the initial value problem $d\boldsymbol{u}/dt = A\boldsymbol{u}$ with $\boldsymbol{u}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^\intercal$.

(c) Let V be the vector space consisting of all vectors $\boldsymbol{v}$ such that the solution $\boldsymbol{u}(t)$ to $d\boldsymbol{u}/dt = A\boldsymbol{u}$ with $\boldsymbol{u}(0) = \boldsymbol{v}$ satisfies $\lim_{t\to\infty} \boldsymbol{u}(t) = \boldsymbol{O}$. Find a basis of V.

Math 218D: Week 12 Discussion

Study Copy

November 11, 2021

Problem 1. What is a spectral factorization? Which matrices have spectral factorizations?

Problem 2. Suppose S is a real-symmetric matrix whose eigenspaces are given by

$$\mathcal{E}_S(-3) = \mathrm{Span}\{\begin{bmatrix}1 & -2 & 0 & 2\end{bmatrix}^\intercal, \begin{bmatrix}-1 & -3 & -2 & 2\end{bmatrix}^\intercal\} \quad \mathcal{E}_S(5) = \mathrm{Span}\{\begin{bmatrix}0 & 2 & -1 & 2\end{bmatrix}^\intercal\} \quad \mathcal{E}_S(9) = ?$$

(a) Find a basis of $\mathcal{E}_S(9)$.

(b) Find a spectral factorization of S.

Problem 3. Show that the following quadratic form is *indefinite*.

$$q(\boldsymbol{x}) = 2\,x_1^2 + 8\,x_1x_2 + 2\,x_2^2 + 2\,x_1x_3 + 34\,x_2x_3 + 3670\,x_3^2 + 2\,x_1x_4 - 2\,x_2x_4 - 10\,x_3x_4 - 4\,x_4^2$$

Hint. We can do this very quickly by plugging in two vectors into $q(\boldsymbol{x})$.

Problem 4. Show that every Gramian is positive semidefinite. Under what condition is a Gramian positive definite?

Problem 5. Consider the spectral factorization

$$\overset{S}{\begin{bmatrix} 2 & 3 & -11 \\ 3 & 10 & -3 \\ -11 & -3 & 2 \end{bmatrix}} = \overset{U}{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}} \overset{D}{\begin{bmatrix} 16 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -9 \end{bmatrix}} \overset{U^*}{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}$$

Let $q(x_1, x_2, x_3)$ be the quadratic form corresponding to S.

(a) $q(x_1, x_2, x_3) = \underline{\qquad}\cdot x_1^2 + \underline{\qquad}\cdot x_2^2 + \underline{\qquad}\cdot x_3^2 + \underline{\qquad}\cdot x_1x_2 + \underline{\qquad}\cdot x_1x_3 + \underline{\qquad}\cdot x_2x_3$

(b) What is the definiteness of S? Select all that apply.

○ pos definite ○ pos semidefinite ○ neg definite ○ neg semidefinite ○ indefinite

(c) To "complete the square" we introduce variables $\{y_1, y_2, y_3\}$. Fill-in the blanks below to express $\{y_1, y_2, y_3\}$ in terms of the variables $\{x_1, x_2, x_3\}$.

$y_1 = \underline{\qquad\qquad\qquad}$ $y_2 = \underline{\qquad\qquad\qquad}$ $y_3 = \underline{\qquad\qquad\qquad}$

Fill-in the blanks below to express $q(x_1, x_2, x_3)$ as a function of $\{y_1, y_2, y_3\}$.

$q(y_1, y_2, y_3) = \underline{\qquad}\cdot y_1^2 + \underline{\qquad}\cdot y_2^2 + \underline{\qquad}\cdot y_3^2 + \underline{\qquad}\cdot y_1y_2 + \underline{\qquad}\cdot y_1y_3 + \underline{\qquad}\cdot y_2y_3$

(d) If possible, find a matrix A such that $S = A^*A$. If this is not possible, then explain why.

Math 218D: Week 13 Discussion

Study Copy

November 18, 2021

Problem 1. What is a singular value decomposition?

Problem 2. Let σ_1, σ_2, and σ_3 be the singular values of

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 2 & -2 & -1 & 0 \\ -3 & -1 & 0 & -1 \end{bmatrix}$$

Calculate $\sigma_1^2 + \sigma_2^2 + \sigma_3^2$.

Problem 3. If possible, construct a 2×2 matrix A with characteristic polynomial $\chi_A(t) = (t-1)^2$ and with exactly one singular value given by $\sigma_1 = 1$. If this is not possible, then explain why.

Problem 4. Find the singular values of $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & -2 \\ 2 & -2 & 2 & 4 \end{bmatrix}$.

Problem 5. Suppose A is 3×3 with exactly two singular values $\sigma_1 = \sqrt{19}$ and $\sigma_2 = 3$. If possible, calculate $\det(A)$. If not, then exmplain why.

Problem 6. Suppose that $A = U\Sigma V^*$ is a singular value decomposition and define $A^+ = V\Sigma^{-1}U^*$. Show that $A^+\boldsymbol{b}$ is a least squares approximate solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Math 218D: Week 15 Discussion

Study Copy

December 2, 2021

Problem 1. Let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{f}(x,y) = \begin{bmatrix} x^2 + \sin(x) \\ x(y-2) \\ -3\,xy + y^2 \end{bmatrix}$$

Use the local linearization of $\boldsymbol{f}$ at the point $P = (0,1)$ to approximate $\boldsymbol{f}(1/2, 1/2)$.

Problem 2. Calculate $S = LDL^{\intercal}$ where $S = \begin{bmatrix} 2 & 12 & 10 \\ 12 & 67 & 105 \\ 10 & 105 & -351 \end{bmatrix}$.

Problem 3. Suppose that factoring $S = LDL^\intercal$ allows us to write the quadratic form $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$ as

$$q(\boldsymbol{x}) = 10\,(x_1 - 5\,x_2 + 2\,x_3)^2 - 11\,(x_2 - 6\,x_3)^2 - 5\,x_3^2$$

Find L and D and determine the definitess of S.

Problem 4. Determine the definiteness of $S = \begin{bmatrix} 0 & 4 & -6 & 8 & 16 \\ 4 & -650 & 3 & 1941 & -1 \\ -6 & 3 & 8 & -144 & 16 \\ 8 & 1941 & -144 & 2 & 18 \\ 16 & -1 & 16 & 18 & 52 \end{bmatrix}$.

Problem 5. Find R the Cholesky factorization $S = R^\intercal R$ of $S = \begin{bmatrix} 9 & 15 & -6 \\ 15 & 29 & 8 \\ -6 & 8 & 206 \end{bmatrix}$.