Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: Solutions

NetID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

September 30, 2022

- There are 100 points and 6 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

(5 pts) **Problem 1.** The matrix $\begin{bmatrix} 3 & 6 & -6 & 15 & 18 \\ -4 & -8 & 8 & -20 & -24 \\ 2 & 4 & -4 & * & 12 \\ 9 & 18 & -18 & 45 & 54 \end{bmatrix}$ has rank one if the entry marked $*$ equals $\underline{\quad 10 \quad}$.

Problem 2. Suppose that the first column of a matrix A is a vector with length three and that A satisfies each of the following equations.

$$\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(5 pts) (*a*) If we write $\mathbb{R}^a \xrightarrow{A} \mathbb{R}^b$, then $a = \underline{\quad 4 \quad}$ and $b = \underline{\quad 3 \quad}$.

(5 pts) (*b*) Calculate $\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. The columns of the second matrix come from the vectors multiplied in by A in the second, first, and third equation respectively. This means that

$$\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 \\ -5 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

(5 pts) (*c*) Could A be the incidence matrix of a directed graph? Briefly explain why or why not.

Solution. The middle equation is telling us that the third column of A is $\begin{bmatrix} 7 & -5 & 2 \end{bmatrix}^\intercal$, but the only valid entries in an incidence matrix are -1, 0, and 1. This means that A *cannot* be the incidence matrix of a digraph.

(5 pts) (*d*) Calculate the inner product of the first two columns of A. *Hint.* The first equation gives a relationship between the first two columns of A.

Solution. The first equation is telling us that $-2 \cdot \mathrm{Col}_1 + \mathrm{Col}_2 = \boldsymbol{O}$, which gives $\mathrm{Col}_2 = 2 \cdot \mathrm{Col}_1$. We are told that $\|\mathrm{Col}_1\| = 3$, so

$$\langle \mathrm{Col}_1, \mathrm{Col}_2 \rangle = \langle \mathrm{Col}_1, 2 \cdot \mathrm{Col}_1 \rangle = 2 \cdot \langle \mathrm{Col}_1, \mathrm{Col}_1 \rangle = 2 \cdot \|\mathrm{Col}_1\|^2 = 2 \cdot 3^2 = 18$$

Problem 3. Suppose that A is a matrix with 2022 columns and that $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal unit vectors satisfying $\boldsymbol{v} \in \mathcal{E}_A(9)$ and $\boldsymbol{w} \in \mathcal{E}_{A^\intercal}(-3)$.

(3 pts) (a) The number of rows of A is __2022__.

(8 pts) (b) Show that $\boldsymbol{u} = A\boldsymbol{v}$ is an eigenvector of A and identify the corresponding eigenvalue.

Solution. We are told that $\boldsymbol{v} \in \mathcal{E}_A(9)$, which means that $A\boldsymbol{v} = 9 \cdot \boldsymbol{v}$. We wish to show that $\boldsymbol{u}$ is an eigenvector of A, which occurs if $A\boldsymbol{u} = \lambda \cdot \boldsymbol{u}$ for some scalar λ. Putting our given information together gives

$$A\boldsymbol{u} = AA\boldsymbol{v} = 9 \cdot A\boldsymbol{v} = 9 \cdot \boldsymbol{u}$$

Evidently, $\boldsymbol{u}$ is indeed an eigenvector of A with eigenvalue $\lambda = 9$.

(8 pts) (c) Calculate $\langle \boldsymbol{v} + A\boldsymbol{w} + \boldsymbol{w}, \boldsymbol{w} \rangle$.

Solution. We are told that $\boldsymbol{v} \in \mathcal{E}_A(9)$ and $\boldsymbol{w} \in \mathcal{E}_{A^\intercal}(-3)$ are orthogonal unit vectors, which means the following equations hold.

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \qquad \|\boldsymbol{v}\| = \|\boldsymbol{w}\| = 1 \qquad A\boldsymbol{v} = 9 \cdot \boldsymbol{v} \qquad A^\intercal \boldsymbol{w} = -3 \cdot \boldsymbol{w}$$

The linearity properties of inner products then gives

$$\begin{aligned} \langle \boldsymbol{v} + A\boldsymbol{w} + \boldsymbol{w}, \boldsymbol{w} \rangle &= \langle \boldsymbol{v}, \boldsymbol{w} \rangle + \langle A\boldsymbol{w}, \boldsymbol{w} \rangle + \langle \boldsymbol{w}, \boldsymbol{w} \rangle \\ &= 0 + \langle \boldsymbol{w}, A^\intercal \boldsymbol{w} \rangle + \|\boldsymbol{w}\|^2 \\ &= \langle \boldsymbol{w}, A^\intercal \boldsymbol{w} \rangle + \|\boldsymbol{w}\|^2 \\ &= \langle \boldsymbol{w}, -3 \cdot \boldsymbol{w} \rangle + 1^2 \\ &= -3 \cdot \langle \boldsymbol{w}, \boldsymbol{w} \rangle + 1 \\ &= -3 \cdot \|\boldsymbol{w}\|^2 + 1 \\ &= -3 \cdot 1^2 + 1 \\ &= -2 \end{aligned}$$

Problem 4. Suppose $R = \begin{bmatrix} 3 & 2 & 5 & 1 \\ 0 & 7 & 9 & 2 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & * & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon form and that $\boldsymbol{b}$ is a vector making $R\boldsymbol{x} = \boldsymbol{b}$ *inconsistent*.

(2 pts) (a) The $(4, 3)$ entry of R (marked $*$) must equal __0__.

(6 pts) (b) $\operatorname{rank}(R) =$ __4__, $\operatorname{nullity}(R) =$ __0__, and $\operatorname{nullity}(R^\intercal) =$ __1__

(5 pts) (c) Consider the augmented matrix $M = [R \mid \boldsymbol{b}]$ (so M is 5×5). Which of the following adjectives applies to M?

√ nonsingular ◯ symmetric ◯ rank one ◯ identity matrix ◯ reduced row echelon form

Problem 5. Consider the nonsingular matrix E and the reduced row echelon form matrix R given by

$$E = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 0 & 10 \\ -2 & 1 & -6 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -7 & 0 & -6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(12 pts) (a) Use the Gauß-Jordan algorithm to calculate the inverse of E.

Solution. Following the algorithm from class, we have

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 3 & 1 & 0 & 0 \\ 3 & 0 & 10 & 0 & 1 & 0 \\ -2 & 1 & -6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\boldsymbol{r}_3 + 2\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_3]{\boldsymbol{r}_2 - 3\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\boldsymbol{r}_2 \leftrightarrow \boldsymbol{r}_3} \left[\begin{array}{rrr|rrr} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\boldsymbol{r}_1 - 3\cdot\boldsymbol{r}_3 \to \boldsymbol{r}_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 10 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right]$$

This gives $E^{-1} = \begin{bmatrix} 10 & -3 & 0 \\ 2 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix}$.

(12 pts) (b) Suppose that A is the matrix satisfying $EA = R$ and consider the vector $\boldsymbol{b} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^\intercal$. Find all solutions to the system $A\boldsymbol{x} = \boldsymbol{b}$.

Solution. Multiplying $A\boldsymbol{x} = \boldsymbol{b}$ from the left by E gives $EA\boldsymbol{x} = E\boldsymbol{b}$, which reduces to $R\boldsymbol{x} = E\boldsymbol{b}$. In augmented form, this system is

$$\left[\begin{array}{rrrr|r} 1 & -7 & 0 & -6 & 1 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent because there is no pivot in the augmented column. The dependent variables are x_1, x_3, and the free variables are $x_2 = c_1$ and $x_4 = c_2$. The two equations in the above system are

$$x_1 - 7\,c_1 - 6\,c_2 = 1 \qquad\qquad x_3 + 4\,c_2 = 3$$

Solving for the dependent variables gives $x_1 = 1 + 7\,c_1 + 6\,c_2$ and $x_3 = 3 - 4\,c_2$. The general solution is then

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7\,c_1 + 6\,c_2 + 1 \\ c_1 \\ -4\,c_2 + 3 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Problem 6. Suppose that A is 4×4 and that the matrix $5 \cdot I_4 - A$ can be reduced to row echelon form U with the following elementary row operations.

$$\left[\; 5 \cdot I_4 - A \;\right] \xrightarrow[]{\begin{array}{l} \boldsymbol{r}_2 - \boldsymbol{r}_1 \to \boldsymbol{r}_2 \\ \boldsymbol{r}_4 - 3 \cdot \boldsymbol{r}_1 \to \boldsymbol{r}_4 \end{array}} \left[\; \begin{array}{c} \text{new} \\ \text{matrix} \end{array} \;\right] \xrightarrow[]{\begin{array}{l} \boldsymbol{r}_3 - \boldsymbol{r}_2 \to \boldsymbol{r}_3 \\ \boldsymbol{r}_4 + 4 \cdot \boldsymbol{r}_2 \to \boldsymbol{r}_4 \end{array}} \left[\; \begin{array}{c} \text{new} \\ \text{matrix} \end{array} \;\right] \xrightarrow{\boldsymbol{r}_3 \leftrightarrow \boldsymbol{r}_4} \overset{U}{\begin{bmatrix} 3 & 2 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

These row reductions give a factorization $P(5 \cdot I_4 - A) = LU$.

(7 pts) (a) Find L.

Solution. According to our procedure from class, we introduce entries into L during elimination steps and swap rows during row swap steps.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & -4 & 0 & 0 \end{bmatrix} \to \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}}$$

(6 pts) (b) Find P.

Solution. According to our algorithm from class, P starts as I_4 and is only affected by row-swaps.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \overset{P}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}$$

(6 pts) (c) The scalar $\lambda = \underline{\quad 5 \quad}$ is an eigenvalue of A and its geometric multiplicity is $\operatorname{gm}_A(\lambda) = \underline{\quad 1 \quad}$.