Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: ____________________ *NetID:* ____________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ____________________

September 30, 2022

- There are 100 points and 6 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

(5 pts) **Problem 1.** The matrix $\begin{bmatrix} 3 & 6 & -6 & 15 & 18 \\ -4 & -8 & 8 & -20 & -24 \\ 2 & 4 & -4 & * & 12 \\ 9 & 18 & -18 & 45 & 54 \end{bmatrix}$ has rank one if the entry marked $*$ equals ________.

Problem 2. Suppose that the first column of a matrix A is a vector with length three and that A satisfies each of the following equations.

$$\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(5 pts) (a) If we write $\mathbb{R}^a \xrightarrow{A} \mathbb{R}^b$, then $a =$ ________ and $b =$ ________.

(5 pts) (b) Calculate $\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(5 pts) (c) Could A be the incidence matrix of a directed graph? Briefly explain why or why not.

(5 pts) (d) Calculate the inner product of the first two columns of A. *Hint.* The first equation gives a relationship between the first two columns of A.

Problem 3. Suppose that A is a matrix with 2022 columns and that $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal unit vectors satisfying $\boldsymbol{v} \in \mathcal{E}_A(9)$ and $\boldsymbol{w} \in \mathcal{E}_{A^\intercal}(-3)$.

(3 pts) (*a*) The number of rows of A is ________.

(8 pts) (*b*) Show that $\boldsymbol{u} = A\boldsymbol{v}$ is an eigenvector of A and identify the corresponding eigenvalue.

(8 pts) (*c*) Calculate $\langle \boldsymbol{v} + A\boldsymbol{w} + \boldsymbol{w}, \boldsymbol{w} \rangle$.

Problem 4. Suppose $R = \begin{bmatrix} 3 & 2 & 5 & 1 \\ 0 & 7 & 9 & 2 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & * & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon form and that $\boldsymbol{b}$ is a vector making $R\boldsymbol{x} = \boldsymbol{b}$ *inconsistent.*

(2 pts) (*a*) The $(4,3)$ entry of R (marked $*$) must equal ________.

(6 pts) (*b*) $\operatorname{rank}(R) =$ ________, $\operatorname{nullity}(R) =$ ________, and $\operatorname{nullity}(R^\intercal) =$ ________

(5 pts) (*c*) Consider the augmented matrix $M = [R \mid \boldsymbol{b}]$ (so M is 5×5). Which of the following adjectives applies to M?

○ nonsingular ○ symmetric ○ rank one ○ identity matrix ○ reduced row echelon form

Problem 5. Consider the nonsingular matrix E and the reduced row echelon form matrix R given by

$$E = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 0 & 10 \\ -2 & 1 & -6 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -7 & 0 & -6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(12 pts) (a) Use the Gauß-Jordan algorithm to calculate the inverse of E.

(12 pts) (b) Suppose that A is the matrix satisfying $EA = R$ and consider the vector $\boldsymbol{b} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^{\mathsf{T}}$. Find all solutions to the system $A\boldsymbol{x} = \boldsymbol{b}$.

Problem 6. Suppose that A is 4×4 and that the matrix $5\cdot I_4 - A$ can be reduced to row echelon form U with the following elementary row operations.

$$\left[\begin{array}{c} \\ 5\cdot I_4 - A \\ \\ \end{array}\right] \xrightarrow[\boldsymbol{r}_4 \;-\; 3\cdot\boldsymbol{r}_1 \;\to\; \boldsymbol{r}_4]{\boldsymbol{r}_2 \;-\; \boldsymbol{r}_1 \;\to\; \boldsymbol{r}_2} \left[\begin{array}{c} \text{new} \\ \text{matrix} \end{array}\right] \xrightarrow[\boldsymbol{r}_4 \;+\; 4\cdot\boldsymbol{r}_2 \;\to\; \boldsymbol{r}_4]{\boldsymbol{r}_3 \;-\; \boldsymbol{r}_2 \;\to\; \boldsymbol{r}_3} \left[\begin{array}{c} \text{new} \\ \text{matrix} \end{array}\right] \xrightarrow{\boldsymbol{r}_3 \leftrightarrow \boldsymbol{r}_4} \overset{U}{\begin{bmatrix} 3 & 2 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

These row reductions give a factorization $P(5\cdot I_4 - A) = LU$.

(7 pts) (a) Find L.

(6 pts) (b) Find P.

(6 pts) (c) The scalar $\lambda =$ _______ is an eigenvalue of A and its geometric multiplicity is $\operatorname{gm}_A(\lambda) =$ _______.