Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: Solutions

NetID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

October 28, 2022

- There are 100 points and 6 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

Problem 1. The rows of $A = \begin{bmatrix} -1 & 1 & 2 \\ -2 & 1 & 5 \\ 0 & 0 & 0 \\ 1 & -3 & 1 \\ -4 & 5 & 6 \end{bmatrix}$ satisfy $\text{Row}_1 + \text{Row}_2 + \text{Row}_3 = \text{Row}_4 + \text{Row}_5$.

(5 pts) (*a*) Determine if $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\intercal \in \text{Null}(A)$. Clearly explain your reasoning.

Solution. This is a question of whether or not $A\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\intercal = \boldsymbol{O}$, which is resolved by the calculation

$$\overset{A}{\begin{bmatrix} -1 & 1 & 2 \\ -2 & 1 & 5 \\ 0 & 0 & 0 \\ 1 & -3 & 1 \\ -4 & 5 & 6 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Evidently, $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\intercal \notin \text{Null}(A)$.

Of course, it also suffices to note that $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\intercal$ is not orthogonal to any of the second, fourth, or fifth rows of A.

(5 pts) (*b*) Determine if $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^\intercal \in \text{Col}(A)$. Clearly explain your reasoning. *Hint.* What is the third coordinate of every column of A?

Solution. The third coordinate of every column of A is zero, which means no linear combination of the columns of A will produce $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^\intercal$. This means that $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^\intercal \notin \text{Col}(A)$.

(5 pts) (*c*) Determine if $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\intercal \in \text{Col}(A^\intercal)$. Clearly explain your reasoning.

Solution. This matrix has three rows, so $\text{Col}(A^\intercal) \subset \mathbb{R}^3$. The vector $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\intercal$ is certainly not in $\text{Col}(A^\intercal)$.

(5 pts) (*d*) Find a vector $\boldsymbol{v}$ in $\text{Null}(A^\intercal)$ such that every coordinate of $\boldsymbol{v}$ is not zero.

Solution. We are told that $\text{Row}_1 + \text{Row}_2 + \text{Row}_3 = \text{Row}_4 + \text{Row}_5$, which is equivalent to

$$\text{Row}_1 + \text{Row}_2 + \text{Row}_3 - \text{Row}_4 - \text{Row}_5 = \boldsymbol{O}$$

This means that

$$\begin{bmatrix} & A^\intercal & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A suitable choice for $\boldsymbol{v}$ is thus $\boldsymbol{v} = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 \end{bmatrix}^\intercal$.

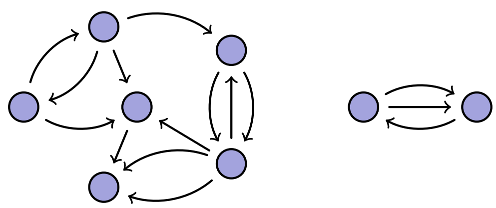
(8 pts) **Problem 2.** Suppose that A is $m \times n$ and that $\boldsymbol{v} \in \operatorname{Null}(A^\intercal A)$. Find the scalar value of $\|A\boldsymbol{v}\|^2$.

Solution. We are given that $\boldsymbol{v} \in \operatorname{Null}(A^\intercal A)$, which means $A^\intercal A\boldsymbol{v} = \boldsymbol{O}$. It follows that

$$\|A\boldsymbol{v}\|^2 = \langle A\boldsymbol{v}, A\boldsymbol{v}\rangle = \langle \boldsymbol{v}, A^\intercal A\boldsymbol{v}\rangle = \langle \boldsymbol{v}, \boldsymbol{O}\rangle = 0$$

Of course, the second equals sign is justified by the adjoint property of inner products.

Problem 3. The directed graph G depicted below has eight nodes and fifteen arrows.



Let A be the incidence matrix of G.

(4 pts) (*a*) $h_0(G) =$ ___2___ and $h_1(G) =$ ___9___

(8 pts) (*b*) Are the rows of A linearly independent? Clearly explain why or why not.

Solution. Since the rows of A are the columns of $A^\intercal$, this is a question of whether or not $A^\intercal$ has full column rank, which is the same as asking if A is full row rank.

Note that A is 8×15. Since $\dim \operatorname{Null}(A^\intercal) = h_0(G) = 2$, it follows that $\operatorname{rank}(A) = \dim \operatorname{Col}(A) = 8 - 2 = 6 \neq 8$. This means that A does not have independent rows!

(8 pts) **Problem 4.** The scalar $\lambda = 3$ is an eigenvalue of $A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ -1 & 4 & 2 & 3 \\ 2 & -2 & -1 & -6 \\ -1 & 1 & 2 & 6 \end{bmatrix}$. According to a theorem from class, every basis of $\mathcal{E}_A(\lambda)$ has the same number of vectors. Find this number and clearly justify your answer.

Solution. This is the dimension of the eigenspace $\mathcal{E}_A(3) = \operatorname{Null}(3 \cdot I_4 - A)$, so we are looking for the nullity of $3 \cdot I_4 - A$. The characteristic matrix is

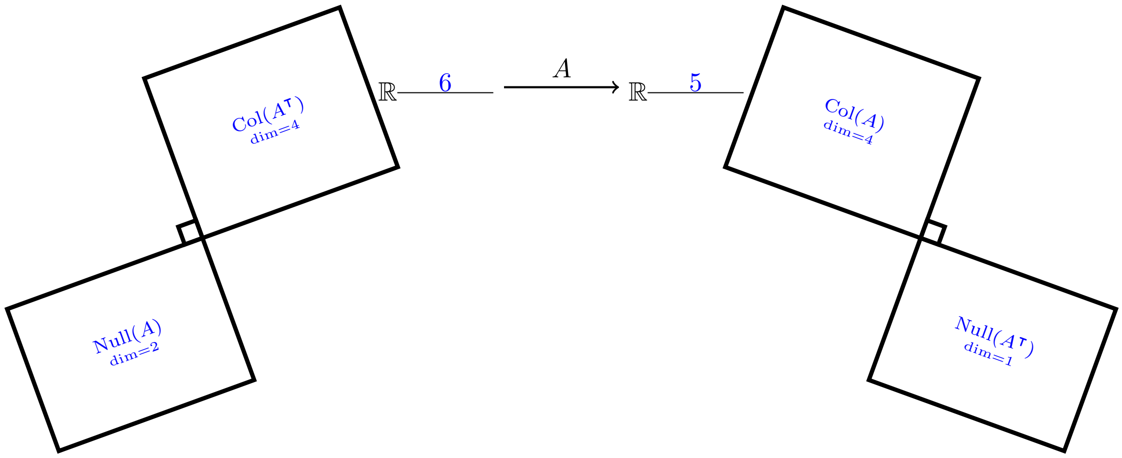
$$\overset{l \cdot I_4}{\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}} - \overset{A}{\begin{bmatrix} 2 & 1 & 2 & 3 \\ -1 & 4 & 2 & 3 \\ 2 & -2 & -1 & -6 \\ -1 & 1 & 2 & 6 \end{bmatrix}} = \overset{\text{rank one!}}{\begin{bmatrix} 1 & -1 & -2 & -3 \\ 1 & -1 & -2 & -3 \\ -2 & 2 & 4 & 6 \\ 1 & -1 & -2 & -3 \end{bmatrix}}$$

Every column of $3 \cdot I_4 - A$ is a multiple of the first column, so $3 \cdot I_4 - A$ is rank one. This means that $\dim \mathcal{E}_A(3) = \operatorname{nullity}(3 \cdot I_4 - A) = 4 - \operatorname{rank}(3 \cdot I_4 - A) = 3$.

Problem 5. Suppose that $EA = R$ where

$$E = \begin{bmatrix} 1 & 2 & 2 & -9 & -17 \\ -2 & -3 & -1 & 10 & 16 \\ 0 & -1 & -2 & 6 & 14 \\ 0 & -1 & -2 & 7 & 16 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 7 \\ 0 & 1 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(10 pts) (*a*) Fill in every missing label in the picture of the four fundamental subspaces below, including the dimension of each fundamental subspace.



(4 pts) (*b*) Which of the following rows of E is orthogonal to the columns of A? Select all that apply (no partial credit on this problem). ○ Row_1 ○ Row_2 ○ Row_3 ○ Row_4 √ Row_5

(4 pts) (*c*) Which of the following vectors belongs to Col(A)? Select all that apply (no partial credit on this problem).
√ $\begin{bmatrix} 7 & -5 & 4 & 0 & 0 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 0 & 0 & 5 & 1 & 0 \end{bmatrix}^\intercal$ √ $\begin{bmatrix} 2 & 0 & 5 & 0 & 0 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 3 & 0 & 0 & 0 & -2 \end{bmatrix}^\intercal$

(4 pts) (*d*) Which of the following vectors belongs to Null($A^\intercal$)? Select all that apply (no partial credit on this problem).
○ $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\intercal$ √ $\begin{bmatrix} 0 & 0 & 0 & 2 & 6 \end{bmatrix}^\intercal$ √ $\begin{bmatrix} 0 & 0 & 0 & -3 & -9 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 0 & 0 & 0 & 3 & -1 \end{bmatrix}^\intercal$

(10 pts) (*e*) Find the projection of $\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 100 & 100 & 100 \end{bmatrix}^\intercal$ onto Col(A).

Solution. Note that $\dim \operatorname{Col}(A) = 4$ while $\dim \operatorname{Null}(A^\intercal) = 1$. Since $\operatorname{Null}(A^\intercal) = \operatorname{Col}(A)^\perp$ it will be easier to start by projecting $\boldsymbol{b}$ onto Null($A^\intercal$).

To do so, note that the last row $\boldsymbol{v}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 3 \end{bmatrix}^\intercal$ of E gives our basis vector of Null($A^\intercal$). The projection of $\boldsymbol{b}$ onto Null($A^\intercal$) is then

$$P_{\operatorname{Null}(A^\intercal)}\boldsymbol{b} = \frac{1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1\boldsymbol{v}_1^\intercal\boldsymbol{b} = \frac{1}{10}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 100 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 40 \\ 120 \end{bmatrix}$$

The projection of $\boldsymbol{b}$ onto Col(A) is then

$$\boldsymbol{b} - P_{\operatorname{Null}(A^\intercal)}\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 100 \\ 100 \\ 100 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 40 \\ 120 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 100 \\ 60 \\ -20 \end{bmatrix}$$

Problem 6. Consider the matrices G, Q, and R given by

$$G = \begin{bmatrix} 1 & -14 & 9 \\ -14 & 221 & -106 \\ 9 & -106 & 146 \end{bmatrix} \qquad Q = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -14 & * \\ 0 & 5 & * \\ 0 & 0 & 7 \end{bmatrix}$$

Suppose that A is a matrix whose Gramian is G and that $A = QR$. The entries of R marked $*$ are unknown.

(5 pts) (*a*) Every column of $I_4 - QQ^\intercal$ belongs to one of the following vector spaces. Select this space.

○ $\text{Col}(A^\intercal)$ ○ $\text{Null}(A)$ ○ $\text{Col}(A)$ ✓ $\text{Null}(A^\intercal)$

(5 pts) (*b*) If possible, calculate $R^\intercal R$. If this is not possible, then explain why.

Solution. In class we argued that $A^\intercal A = R^\intercal R$ when $A = QR$. We are told that $A^\intercal A = G$, so we immediately know that $R^\intercal R = G$ too.

(10 pts) (*c*) The vector $\boldsymbol{b} = \begin{bmatrix} 5\sqrt{3} & -5\sqrt{3} & 0 & 0 \end{bmatrix}^\intercal$ satisfies $A^\intercal \boldsymbol{b} = \begin{bmatrix} 5 & -95 & 25 \end{bmatrix}^\intercal$. Find all solutions $\boldsymbol{x}$ to $G\boldsymbol{x} = A^\intercal \boldsymbol{b}$.

Solution. The system $G\boldsymbol{x} = A^\intercal \boldsymbol{b}$ is $A^\intercal A\boldsymbol{x} = A^\intercal \boldsymbol{b}$, which is the least squares problem! Since $A = QR$, we can instead solve $R\boldsymbol{x} = Q^\intercal \boldsymbol{b}$.

To do so, we start by calculating

$$\frac{1}{\sqrt{3}} \overset{Q^\intercal}{\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}} \overset{\boldsymbol{b}}{\begin{bmatrix} 5\sqrt{3} \\ -5\sqrt{3} \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix}$$

Now, we solve $R\boldsymbol{x} = Q^\intercal \boldsymbol{b}$ with back-substitution

$$\begin{array}{rcrcrcrcl} x_1 & - & 14\,x_2 & + & *\,x_3 & = & 5 & \to & x_1 = -9 \\ & & 5\,x_2 & + & *\,x_3 & = & -5 & \to & x_2 = -1 \\ & & & & 7\,x_3 & = & 0 & \to & x_3 = 0 \end{array}$$

The only solution to $G\boldsymbol{x} = A^\intercal \boldsymbol{b}$ is $\boldsymbol{x} = \begin{bmatrix} -9 & -1 & 0 \end{bmatrix}^\intercal$.