Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: ____________________ *NetID:* ____________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ____________________

October 28, 2022

- There are 100 points and 6 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. The rows of $A = \begin{bmatrix} -1 & 1 & 2 \\ -2 & 1 & 5 \\ 0 & 0 & 0 \\ 1 & -3 & 1 \\ -4 & 5 & 6 \end{bmatrix}$ satisfy $\text{Row}_1 + \text{Row}_2 + \text{Row}_3 = \text{Row}_4 + \text{Row}_5$.

(5 pts) (*a*) Determine if $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\intercal \in \text{Null}(A)$. Clearly explain your reasoning.

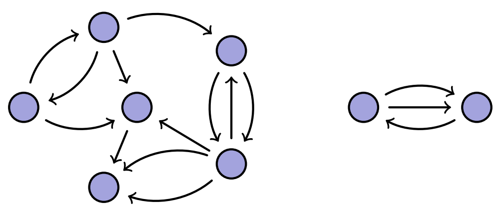
(5 pts) (*b*) Determine if $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^\intercal \in \text{Col}(A)$. Clearly explain your reasoning. *Hint.* What is the third coordinate of every column of A?

(5 pts) (*c*) Determine if $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\intercal \in \text{Col}(A^\intercal)$. Clearly explain your reasoning.

(5 pts) (*d*) Find a vector $\boldsymbol{v}$ in $\text{Null}(A^\intercal)$ such that every coordinate of $\boldsymbol{v}$ is not zero.

(8 pts) **Problem 2.** Suppose that A is $m \times n$ and that $\boldsymbol{v} \in \operatorname{Null}(A^\intercal A)$. Find the scalar value of $\|A\boldsymbol{v}\|^2$.

Problem 3. The directed graph G depicted below has eight nodes and fifteen arrows.



Let A be the incidence matrix of G.

(4 pts) (a) $h_0(G) =$ ______________ and $h_1(G) =$ ______________

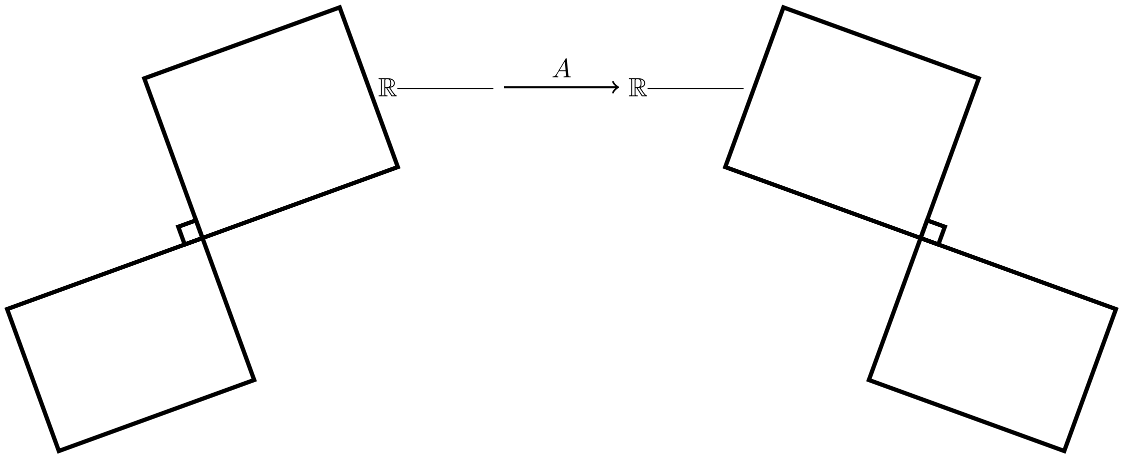
(8 pts) (b) Are the rows of A linearly independent? Clearly explain why or why not.

(8 pts) **Problem 4.** The scalar $\lambda = 3$ is an eigenvalue of $A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ -1 & 4 & 2 & 3 \\ 2 & -2 & -1 & -6 \\ -1 & 1 & 2 & 6 \end{bmatrix}$. According to a theorem from class, every basis of $\mathcal{E}_A(\lambda)$ has the same number of vectors. Find this number and clearly justify your answer.

Problem 5. Suppose that $EA = R$ where

$$E = \begin{bmatrix} 1 & 2 & 2 & -9 & -17 \\ -2 & -3 & -1 & 10 & 16 \\ 0 & -1 & -2 & 6 & 14 \\ 0 & -1 & -2 & 7 & 16 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 7 \\ 0 & 1 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(10 pts) (a) Fill in every missing label in the picture of the four fundamental subspaces below, including the dimension of each fundamental subspace.



(4 pts) (b) Which of the following rows of E is orthogonal to the columns of A? Select all that apply (no partial credit on this problem). ○ Row_1 ○ Row_2 ○ Row_3 ○ Row_4 ○ Row_5

(4 pts) (c) Which of the following vectors belongs to $\operatorname{Col}(A)$? Select all that apply (no partial credit on this problem).
○ $\begin{bmatrix} 7 & -5 & 4 & 0 & 0 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 0 & 0 & 5 & 1 & 0 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 2 & 0 & 5 & 0 & 0 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 3 & 0 & 0 & 0 & -2 \end{bmatrix}^\intercal$

(4 pts) (d) Which of the following vectors belongs to $\operatorname{Null}(A^\intercal)$? Select all that apply (no partial credit on this problem).
○ $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 0 & 0 & 0 & 2 & 6 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 0 & 0 & 0 & -3 & -9 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 0 & 0 & 0 & 3 & -1 \end{bmatrix}^\intercal$

(10 pts) (e) Find the projection of $\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 100 & 100 & 100 \end{bmatrix}^\intercal$ onto $\operatorname{Col}(A)$.

Problem 6. Consider the matrices G, Q, and R given by

$$G = \begin{bmatrix} 1 & -14 & 9 \\ -14 & 221 & -106 \\ 9 & -106 & 146 \end{bmatrix} \qquad Q = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -14 & * \\ 0 & 5 & * \\ 0 & 0 & 7 \end{bmatrix}$$

Suppose that A is a matrix whose Gramian is G and that $A = QR$. The entries of R marked $*$ are unknown.

(5 pts) (*a*) Every column of $I_4 - QQ^\intercal$ belongs to one of the following vector spaces. Select this space.

○ $\operatorname{Col}(A^\intercal)$ ○ $\operatorname{Null}(A)$ ○ $\operatorname{Col}(A)$ ○ $\operatorname{Null}(A^\intercal)$

(5 pts) (*b*) If possible, calculate $R^\intercal R$. If this is not possible, then explain why.

(10 pts) (*c*) The vector $\boldsymbol{b} = \begin{bmatrix} 5\sqrt{3} & -5\sqrt{3} & 0 & 0 \end{bmatrix}^\intercal$ satisfies $A^\intercal \boldsymbol{b} = \begin{bmatrix} 5 & -95 & 25 \end{bmatrix}^\intercal$. Find all solutions $\boldsymbol{x}$ to $G\boldsymbol{x} = A^\intercal \boldsymbol{b}$.