Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: Solutions

NetID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

December 2, 2022

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the matrix A and its characteristic polynomial $\chi_A(t)$ given by

$$A = \begin{bmatrix} 1 & 27 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 8 & 3 & c \\ 0 & 0 & -4 & -1 & -4 \end{bmatrix} \qquad \chi_A(t) = t^5 + \underline{\quad -1 \quad} t^4 - 5\,t^3 + 5\,t^2 + 4\,t - 4$$

Note that the $(4,5)$ entry of A is a variable marked c. Also note that the coefficient of t^4 in $\chi_A(t)$ is blank.

(4 pts) (*a*) Fill in the blank coefficient of t^4 in $\chi_A(t)$.

(10 pts) (*b*) Find c. *Hint.* What is $\det(A)$?

Solution. This A is 5×5, so $\det(A) = (-1)^5 \cdot \chi(0) = 4$. On the other hand, we have the direct calculation

$$\overset{\det(A)}{\begin{vmatrix} 1 & 27 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 8 & 3 & c \\ 0 & 0 & -4 & -1 & -4 \end{vmatrix}} \xlongequal[\boldsymbol{r}_5 - \boldsymbol{r}_3 \to \boldsymbol{r}_5]{\boldsymbol{r}_4 + 2\cdot\boldsymbol{r}_3 \to \boldsymbol{r}_4} \begin{vmatrix} 1 & 27 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & -1 & c-8 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

$$\xlongequal[\boldsymbol{r}_5 - \boldsymbol{r}_3 \to \boldsymbol{r}_5]{\boldsymbol{r}_4 + 2\cdot\boldsymbol{r}_3 \to \boldsymbol{r}_4} \begin{vmatrix} 1 & 27 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & -1 & c-8 \\ 0 & 0 & 0 & 0 & c-8 \end{vmatrix}$$

$$= 2\,(c-8)$$

Putting this together gives $2\,(c-8) = 4$, which implies $c = 10$.

(10 pts) (*c*) The scalar 2 is an eigenvalue of A. Suppose that v is an eigenvector of A corresponding to the eigenvalue 2. Show that $\boldsymbol{v}$ is also an eigenvector of $\operatorname{adj}(A)$ and identify its corresponding eigenvalue λ.

Solution. We are told that $A\boldsymbol{v} = 2 \cdot \boldsymbol{v}$. We wish to show that $\operatorname{adj}(A)\boldsymbol{v} = \lambda \cdot \boldsymbol{v}$. To do so, we calculate

$$\begin{aligned} \operatorname{adj}(A)\boldsymbol{v} &= \operatorname{adj}(A)\frac{2}{2}\boldsymbol{v} \\ &= \frac{1}{2}\operatorname{adj}(A)2 \cdot \boldsymbol{v} \\ &= \frac{1}{2}\operatorname{adj}(A)A\boldsymbol{v} \\ &= \frac{1}{2}\det(A) \cdot I_5\boldsymbol{v} \\ &= \frac{\det(A)}{2} \cdot \boldsymbol{v} \\ &= \frac{4}{2} \cdot \boldsymbol{v} \\ &= 2 \cdot \boldsymbol{v} \end{aligned}$$

This shows that $\boldsymbol{v}$ is indeed an eigenvector of $\operatorname{adj}(A)$ with corresponding eigenvalue $\lambda = 2$.

Problem 2. Consider the following matrix factorization $A = XDX^{-1}$.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \overset{X}{\begin{bmatrix} 1 & -3 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ -2 & 2 & 1 & 1 \\ 2 & -4 & 1 & 0 \end{bmatrix}} \overset{D}{\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1-i & 0 \\ 0 & 0 & 0 & 1-i \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} -7 & -9 & -2 & 2 \\ -4 & -5 & -1 & 1 \\ -2 & -2 & 0 & 1 \\ -4 & -6 & -1 & 1 \end{bmatrix}}$$

(5 pts) (*a*) The eigenvalue of A with the largest geometric multiplicity is $\lambda =$ __$1-i$__ and that multiplicity is __2__.

(4 pts) (*b*) $$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} -3i \\ i \\ 2i \\ -4i \end{bmatrix}$$

(8 pts) (*c*) Calculate $\det(A)$. Simplify your answer to a complex number of the form $a + b\,i$.

Solution. This is a diagonalization, so

$$\det(A) = \det(D) = 3\,i(1-i)^2 = 3\,i(1 - 2\,i + i^2) = 3\,i(1 - 2\,i - 1) = 3\,i(-2\,i) = -6\,i^2 = 6$$

(8 pts) (*d*) Calculate $\chi_A(3-i)$. Simplify your answer to a complex number of the form $a + b\,i$.

Solution. The characteristic polynomial of A factors as

$$\chi_A(t) = (t-3)(t-i)(t-(1-i))^2$$

It follows that

$$\begin{aligned} \chi_A(1+i) &= (3-i-3)(3-i-i)((3-i)-(1-i))^2 \\ &= (-i)(3-2\,i)(3-i-1+i)^2 \\ &= (-i)(3-2\,i)2^2 \\ &= (-3\,i+2\,i^2)2^2 \\ &= (-3\,i-2)2^2 \\ &= -8-12\,i \end{aligned}$$

Problem 3. Consider the factorization $A = XBX^{-1}$ below.

$$\overset{A}{\begin{bmatrix} -4 & 2 & -7 \\ -2 & 1 & -5 \\ 2 & -1 & 3 \end{bmatrix}} = \overset{X}{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 0 & -1 & -2 \end{bmatrix}} \overset{B}{\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} -3 & 2 & -4 \\ 4 & -2 & 5 \\ -2 & 1 & -3 \end{bmatrix}} \quad B^2 = \underline{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \quad B^3 = \underline{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

(3 pts) (*a*) Fill in the blanks above to calculate B^2 and B^3.

(4 pts) (*b*) $\operatorname{trace}(A) =$ ___0___, $\det(A) =$ ___0___, and $\chi_A(t) =$ ___t^3___

(10 pts) (*c*) Use the Taylor series definition of matrix exponentials to show that $\exp(Bt) = \begin{bmatrix} 1 & t & 1/2\,t^2 + 2\,t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. The key insight here is that $B^3 = \boldsymbol{O}$, which means that $B^k = \boldsymbol{O}$ for all $k \geq 3$. The Taylor series definition of matrix exponentials then gives

$$\begin{aligned} \exp(Bt) &= \sum_{k=0}^{\infty} \frac{1}{k!} B^k t^k \\ &= I_3 + B\,t + \frac{1}{2} B^2\, t^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 2\,t \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1/2\,t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & t & 1/2\,t^2 + 2\,t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(10 pts) (*d*) Let $\boldsymbol{u}(t)$ be the solution to $\boldsymbol{u}' = A\boldsymbol{u}$ with initial condition $\boldsymbol{u}(0) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\intercal$. Calculate $\boldsymbol{u}(2)$.

Solution. Here, $\boldsymbol{u}(t) = \exp(At)\boldsymbol{u}_0 = X \exp(Bt) X^{-1} \boldsymbol{u}_0$. So, we calculate

$$\begin{aligned} \boldsymbol{u}(2) &= \overset{X}{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 0 & -1 & -2 \end{bmatrix}} \overset{\exp(2\,B)}{\begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} -3 & 2 & -4 \\ 4 & -2 & 5 \\ -2 & 1 & -3 \end{bmatrix}} \overset{\boldsymbol{u}_0}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \\ &= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -5 \\ -5 \\ 2 \end{bmatrix} \end{aligned}$$

Problem 4. Consider the quadratic form $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x}\rangle$ where S is the ***singular*** real-symmetric matrix satisfying

$$\begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix}\overset{\boldsymbol{v}_1}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} = \begin{bmatrix}0\\3\\0\\0\end{bmatrix} \qquad \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix}\overset{\boldsymbol{v}_2}{\begin{bmatrix}-1\\3\\-1\\c\end{bmatrix}} = \begin{bmatrix}-3\\9\\-3\\3c\end{bmatrix} \qquad \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix}\overset{\boldsymbol{v}_3}{\begin{bmatrix}1\\0\\1\\2\end{bmatrix}} = \begin{bmatrix}12\\0\\12\\24\end{bmatrix}$$

Note that the last coordinate of the vector $\boldsymbol{v}_2$ above is marked as the variable c.

(7 pts) (a) The trace of S is $\operatorname{trace}(S) =$ ___18___ and the value of c is $c =$ ___1___.

(3 pts) (b) Which of the following adjectives apply to $q(\boldsymbol{x})$? Select all that apply (no partial credit on this problem).
◯ positive definite √ positive semidefinite ◯ negative definite ◯ negative semidefinite ◯ indefinite

(4 pts) (c) Which of the following vectors $\boldsymbol{x}$ satisfies $q(\boldsymbol{x}) = 0$?
√ $\boldsymbol{x} = \begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}^\intercal$ ◯ $\boldsymbol{x} = \begin{bmatrix}1 & 1 & -1 & 0\end{bmatrix}^\intercal$ ◯ $\boldsymbol{x} = \begin{bmatrix}0 & 3 & -2 & 1\end{bmatrix}^\intercal$ ◯ none of these

(6 pts) (d) Complete the square to write $q(\boldsymbol{x})$ as a linear combination of squares. *Note.* You may leave c as a variable to solve this problem.

Solution. The given equations are $S\boldsymbol{v}_1 = 3\cdot\boldsymbol{v}_1$, $S\boldsymbol{v}_2 = 3\cdot\boldsymbol{v}_2$, and $S\boldsymbol{v}_3 = 12\cdot\boldsymbol{v}_3$. This means that 3 and 12 are eigenvalues of S. We are also told that S is singular, so 0 must also be an eigenvalue of S.
More succinctly, we have

$$\mathcal{E}_S(3) = \operatorname{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \qquad \mathcal{E}_S(12) = \operatorname{Span}\{\boldsymbol{v}_3\} \qquad \mathcal{E}_S(0) = \operatorname{Span}\{\boldsymbol{v}_4\}$$

A spectral factorization would give

$$q(\boldsymbol{x}) = 3\cdot y_1^2 + 3\cdot y_2^2 + 12\cdot y_3^2 + 0\cdot y_4^2 = 3\cdot y_1^2 + 3\cdot y_2^2 + 12\cdot y_3^2$$

So, we only need formulas for y_1, y_2, and y_3. To do so, we need orthonormal bases for $\mathcal{E}_S(3)$ and $\mathcal{E}_S(12)$. Applying Gram-Schmidt to the basis of $\mathcal{E}_S(3)$ gives $\boldsymbol{w}_1 = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}^\intercal$ and then

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_2) = \begin{bmatrix}-1 & 3 & -1 & c\end{bmatrix}^\intercal - \begin{bmatrix}0 & 3 & 0 & 0\end{bmatrix}^\intercal = \begin{bmatrix}-1 & 0 & -1 & c\end{bmatrix}^\intercal$$

The orthonormal bases are

$$\mathcal{E}_S(3) = \operatorname{Span}\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \frac{1}{\sqrt{2+c^2}}\begin{bmatrix}-1\\0\\-1\\c\end{bmatrix}\right\} \qquad \mathcal{E}_S(12) = \operatorname{Span}\left\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\0\\1\\2\end{bmatrix}\right\}$$

Our formulas for y_1, y_2, and y_3 are then

$$y_1 = x_2 \qquad y_2 = \frac{-x_1 - x_3 + c\,x_4}{\sqrt{2+c^2}} \qquad y_3 = \frac{x_1 + x_3 + 2\,x_4}{\sqrt{6}}$$

(4 pts) (e) The quadratic form $q(\boldsymbol{x}) = q(x_1, x_2, x_3, x_4)$ is a scalar field on $\mathbb{R}^4$. Calculate $\dfrac{\partial q}{\partial x_2}$.

Solution. The cool part now is that y_2 and y_3 have no x_2-dependence!
Our partial derivative ends up becoming

$$\frac{\partial q}{\partial x_2} = 2\cdot 3\,y_1\frac{\partial y_1}{\partial x_2} + 2\cdot 3\,y_2\frac{\partial y_2}{\partial x_2} + 2\cdot 12\,y_2\frac{\partial y_3}{\partial x_2} = 6\,y_1\cdot 1 + 6\,y_2\cdot 0 + 24\,y_3\cdot 0 = 6\,x_2$$