Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: ______________________ *NetID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

December 2, 2022

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the matrix A and its characteristic polynomial $\chi_A(t)$ given by

$$A = \begin{bmatrix} 1 & 27 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 8 & 3 & c \\ 0 & 0 & -4 & -1 & -4 \end{bmatrix} \qquad \chi_A(t) = t^5 + \underline{\hspace{2cm}}\, t^4 - 5\,t^3 + 5\,t^2 + 4\,t - 4$$

Note that the $(4,5)$ entry of A is a variable marked c. Also note that the coefficient of t^4 in $\chi_A(t)$ is blank.

(4 pts) (*a*) Fill in the blank coefficient of t^4 in $\chi_A(t)$.

(10 pts) (*b*) Find c. *Hint.* What is $\det(A)$?

(10 pts) (*c*) The scalar 2 is an eigenvalue of A. Suppose that v is an eigenvector of A corresponding to the eigenvalue 2. Show that $\boldsymbol{v}$ is also an eigenvector of $\operatorname{adj}(A)$ and identify its corresponding eigenvalue λ.

Problem 2. Consider the following matrix factorization $A = XDX^{-1}$.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \overset{X}{\begin{bmatrix} 1 & -3 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ -2 & 2 & 1 & 1 \\ 2 & -4 & 1 & 0 \end{bmatrix}} \overset{D}{\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1-i & 0 \\ 0 & 0 & 0 & 1-i \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} -7 & -9 & -2 & 2 \\ -4 & -5 & -1 & 1 \\ -2 & -2 & 0 & 1 \\ -4 & -6 & -1 & 1 \end{bmatrix}}$$

(5 pts) (*a*) The eigenvalue of A with the largest geometric multiplicity is $\lambda =$ _______ and that multiplicity is _______.

(4 pts) (*b*) $\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ 2 \\ -4 \end{bmatrix} =$

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(8 pts) (*c*) Calculate $\det(A)$. Simplify your answer to a complex number of the form $a + b\,i$.

(8 pts) (*d*) Calculate $\chi_A(3 - i)$. Simplify your answer to a complex number of the form $a + b\,i$.

Problem 3. Consider the factorization $A = XBX^{-1}$ below.

$$\underset{A}{\begin{bmatrix} -4 & 2 & -7 \\ -2 & 1 & -5 \\ 2 & -1 & 3 \end{bmatrix}} = \underset{X}{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 0 & -1 & -2 \end{bmatrix}} \underset{B}{\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} \underset{X^{-1}}{\begin{bmatrix} -3 & 2 & -4 \\ 4 & -2 & 5 \\ -2 & 1 & -3 \end{bmatrix}} \qquad B^2 = \underline{\hspace{4cm}} \qquad B^3 = \underline{\hspace{4cm}}$$

(3 pts) (a) Fill in the blanks above to calculate B^2 and B^3.

(4 pts) (b) $\operatorname{trace}(A) =$ ________, $\det(A) =$ ________, and $\chi_A(t) =$ ________________

(10 pts) (c) Use the Taylor series definition of matrix exponentials to show that $\exp(Bt) = \begin{bmatrix} 1 & t & {}^1\!/_2\, t^2 + 2\,t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$.

(10 pts) (d) Let $\boldsymbol{u}(t)$ be the solution to $\boldsymbol{u}' = A\boldsymbol{u}$ with initial condition $\boldsymbol{u}(0) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\intercal$. Calculate $\boldsymbol{u}(2)$.

Problem 4. Consider the quadratic form $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$ where S is the ***singular*** real-symmetric matrix satisfying

$$\begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \overset{\boldsymbol{v}_1}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \overset{\boldsymbol{v}_2}{\begin{bmatrix} -1 \\ 3 \\ -1 \\ c \end{bmatrix}} = \begin{bmatrix} -3 \\ 9 \\ -3 \\ 3\,c \end{bmatrix} \qquad \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \overset{\boldsymbol{v}_3}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 12 \\ 0 \\ 12 \\ 24 \end{bmatrix}$$

Note that the last coordinate of the vector $\boldsymbol{v}_2$ above is marked as the variable c.

(7 pts) (*a*) The trace of S is $\operatorname{trace}(S) =$ ______________ and the value of c is $c =$ ________.

(3 pts) (*b*) Which of the following adjectives apply to $q(\boldsymbol{x})$? Select all that apply (no partial credit on this problem).

○ positive definite ○ positive semidefinite ○ negative definite ○ negative semidefinite ○ indefinite

(4 pts) (*c*) Which of the following vectors $\boldsymbol{x}$ satisfies $q(\boldsymbol{x}) = 0$?

○ $\boldsymbol{x} = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^\intercal$ ○ $\boldsymbol{x} = \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}^\intercal$ ○ $\boldsymbol{x} = \begin{bmatrix} 0 & 3 & -2 & 1 \end{bmatrix}^\intercal$ ○ none of these

(6 pts) (*d*) Complete the square to write $q(\boldsymbol{x})$ as a linear combination of squares. *Note.* You may leave c as a variable to solve this problem.

(4 pts) (*e*) The quadratic form $q(\boldsymbol{x}) = q(x_1, x_2, x_3, x_4)$ is a scalar field on $\mathbb{R}^4$. Calculate $\dfrac{\partial q}{\partial x_2}$.