Matrices and Vectors

Name: ______ NetID: _____

Problem 1. Suppose that $x, y, z \in \mathbb{R}$ are fixed scalars and consider the equation

$$c_1 \cdot \begin{bmatrix} 1\\0\\0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0\\1\\0 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0\\0\\1 \end{bmatrix} = \begin{bmatrix} x\\y\\z \end{bmatrix}$$

What are the only valid choices for c_1, c_2, c_3 that make this equation valid? You are always expected to clearly (and concisely) explain your reasoning.

Problem 2. Suppose that $x, y, z \in \mathbb{R}$ are fixed scalars and consider the equation

$$c_1 \cdot \begin{bmatrix} 1\\0\\0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0\\1\\0 \end{bmatrix} = \begin{bmatrix} x\\y\\z \end{bmatrix}$$

Under what condition on x, y, and z is this equation possible and what are the only valid choices for $c_1 c_2$ if this condition is met?

Understanding how to efficiently demonstrate that equations hold will be reoccuring and important theme in this course. The next three problems are designed to help train you to efficiently work with matrix equations. Make sure to take the time to properly understand how to do these problems!

Problem 3. Suppose that $S = A + A^{\intercal}$ where A is 2022×2022 . Show that S is symmetric. This can be done very quickly by filling in the blanks below.

 $S^{\mathsf{T}} = (A + A^{\mathsf{T}})^{\mathsf{T}} = ____= ___= ___=$

Problem 4. Suppose that $S = S_1 + S_2$ where S_1 and S_2 are 2022×2022 symmetric matrices. Show that S is also symmetric. Again, this can be done very quickly by filling in the blanks below.

 $S^{\mathsf{T}} = (S_1 + S_2)^{\mathsf{T}} = ____= ___=$

Problem 5. We call a matrix X traceless if trace(X) = 0. Suppose that $A = c_1 \cdot A_1 + c_2 \cdot A_2$ where A_1 and A_2 are traceless.

(a) The set-up of this problem is telling us that $\operatorname{trace}(A_1) = \underline{\qquad}$ and $\operatorname{trace}(A_2) = \underline{\qquad}$.

- (b) Trace is a linear operation, which means that $\operatorname{trace}(c_1 \cdot A_1 + c_2 \cdot A_2) = _$
- (c) Show that A is traceless. Like in the preceeding two problems, this can (and absolutely should) be done quickly with a single equation.

Problem 6. It is known that the columns of the matrix A below satisfy the equation on the right.

$$A = \begin{bmatrix} 1 & -2 & 2 & * \\ 0 & 1 & 0 & * \\ 0 & 2 & 1 & * \end{bmatrix} \qquad 2 \cdot \operatorname{Col}_1 - \operatorname{Col}_2 + 3 \cdot \operatorname{Col}_3 - \operatorname{Col}_4 = \boldsymbol{O}$$

Use this information to find the missing column of A.

Matrix-Vector Products

Name: ______ NetID: _____

Problem 1. Consider the following linear combination of vectors

$$-13 \cdot \begin{bmatrix} 1\\-1\\2 \end{bmatrix} + 5 \cdot \begin{bmatrix} 0\\9\\3 \end{bmatrix} + 15 \cdot \begin{bmatrix} 1\\-1\\0 \end{bmatrix} + (-11) \cdot \begin{bmatrix} 0\\-2\\1 \end{bmatrix} + 9 \cdot \begin{bmatrix} 0\\-1\\0 \end{bmatrix}$$

Write this linear combination as a matrix-vector product Av. What is the size of A?

Problem 2. Suppose A is 2022×3 . What vector \boldsymbol{v} would we multiply A by if we wanted to subtract four times the third column of A from the first column of A? Explain.

Problem 3. Suppose A is 3×2022 . What matrix-vector product subtracts the third row of A from the first row of A? *Hint*. The rows of A are the columns of what matrix?

Problem 4. Suppose that A is 4×7 and that $v \in \mathbb{R}^7$. Define w = Av. If B is a matrix such that $Bw \in \mathbb{R}^5$, then what is the size of B?

Problem 5. Suppose that A is $n \times n$ and that $v_1, v_2 \in \mathcal{E}_A(\lambda)$. Show that $v = c_1 \cdot v_1 + c_2 \cdot v_2 \in \mathcal{E}_A(\lambda)$. This can be quickly done by filling in the blanks below.

 $A\boldsymbol{v} = A(c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2) = \underline{\qquad} = \underline{\qquad} = \underline{\qquad}$

Problem 6. Suppose that A is $n \times n$ and that $v \in \mathcal{E}_A(\lambda)$. Calculate $X_\lambda v$ where $X_\lambda = \lambda \cdot I_n - A$.

Problem 7. Consider the matrix A and the vector v given by

$$A = \begin{bmatrix} 1 & 2 & -5 & -3 \\ -4 & -8 & 20 & * \\ -1 & -2 & 5 & * \\ 0 & 0 & 0 & * \end{bmatrix} \qquad \qquad \mathbf{v} = \begin{bmatrix} 1 \\ v_2 \\ -1 \\ 0 \end{bmatrix}$$

Note that both A and v are missing entries!

(a) If we assume that A has rank one, then what are the missing entries of A?

(b) If we assume that v is an eigenvector of A, then there are exactly two possibilities for the missing entry v_2 of v. Find the two possible values of v_2 . In each case, what is the corresponding eigenvalue?

Digraphs

Problem Set

Problem 1. Let G be the weighted digraph given by



(a) Use a matrix-vector product to calculate the weighted net flow through all of the nodes of G. Do the calculation on a computer, but show the matrix and the vector being multiplied.

(b) Let v be the cycle vector corresponding to the cycle (a_6, a_8, a_5, a_4) . Calculate Av where A is the incidence matrix of G. Do the calculation on a computer, but show the matrix and the vector being multiplied. Note. The weight vector in this part of the problem is completely different than the weight vector in part (a). Here, v is a cycle vector, so every entry should either equal 0, 1, or -1.

Problem 2. Let G be the weighted digraph given by



Note that the two weights w_1 and w_3 are unknown.

(a) Suppose that the weighted net flows through the nodes are given by the vector $\boldsymbol{b} = \begin{bmatrix} 18 & -10 & -8 \end{bmatrix}^{\mathsf{T}}$. Find the missing weights w_1 and w_3 .

(b) Is it possible that the weighted net flows through the nodes could instead be given by $\boldsymbol{b} = \begin{bmatrix} 2 & 2 & -1 \end{bmatrix}^{\mathsf{T}}$? Explain why or why not.

Vector Geometry

Name: ______ NetID: _____

Problem 1. Suppose ||v|| = 4 and ||w|| = 7. Use inner products to determine the largest and smallest possible values of ||6v - 5w||. *Hint.* Start by expanding $||6v - 5w||^2$.

Problem 2. Suppose that w_1 and w_2 are orthogonal to a vector v. Show that every linear combination $c_1 \cdot w_1 + c_2 \cdot w_2$ is also orthogonal to v.

Problem 3. Suppose that v_1 and v_2 are two vectors in \mathbb{R}^n where $v_1 \neq 0$. Let

$$oldsymbol{w} = oldsymbol{v}_2 - rac{\langleoldsymbol{v}_2,oldsymbol{v}_1
angle}{\langleoldsymbol{v}_1,oldsymbol{v}_1
angle} \cdot oldsymbol{v}_1$$

Show that \boldsymbol{w} is orthogonal to \boldsymbol{v}_1 .

Problem 4. Consider the vectors \boldsymbol{v} nad \boldsymbol{w} given by

$$\boldsymbol{v} = \begin{bmatrix} -3 & 1 & 0 \end{bmatrix}^\mathsf{T}$$
 $\boldsymbol{w} = \begin{bmatrix} -1 & -2 & -23 \end{bmatrix}^\mathsf{T}$

Suppose that A is a 3 × 3 symmetric matrix satisfying $A\boldsymbol{v} = \begin{bmatrix} -4 & 10 & 8 \end{bmatrix}^{\mathsf{T}}$. Find $\langle \boldsymbol{v}, A\boldsymbol{w} \rangle$.

Matrix Multiplication

Name: ______ NetID: _____

Note. You should always feel free to outsource matrix multiplication calculations to a computer. However, you must always make it clear which matrices you are multiplying in your work.

Problem 1. Consider the vectors \boldsymbol{v} and \boldsymbol{w} given by



Here, we will think of \boldsymbol{v} as a 3 × 1 matrix and \boldsymbol{w} as a 4 × 1 matrix. The *outer product* of \boldsymbol{v} and \boldsymbol{w} is defined by the formula $\boldsymbol{v} \otimes \boldsymbol{w} = \boldsymbol{v} \boldsymbol{w}^{\mathsf{T}}$.

(a) Calculate $v \otimes w$ (note that this can be quickly done without a computer). What type of object is $v \otimes w$?

(b) Are $\boldsymbol{v} \otimes \boldsymbol{w}$ and $\boldsymbol{w} \otimes \boldsymbol{v}$ equal? If not, are they related?

Problem 2. Consider the matrices U and V given by

Let u_1, u_2, u_3 be the columns of U and let v_1, v_2, v_3 be the columns of V.

(a) Calculate $A = u_1 \otimes v_1 + u_2 \otimes v_2 + u_3 \otimes v_3$.

(b) Calculate $B = UV^{\intercal}$. How does the matrix A from part (a) compare to B?

Problem 3. Suppose that A and B are $n \times n$ symmetric matrices and define S = ABA. Show that S is also symmetric.

Problem 4. Suppose that A is 2022×3 , so A looks like

$$A = \left[\begin{array}{ccc} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{array} \right]$$

Suppose that the columns a_1, a_2, a_3 are all orthogonal to each other and that $||a_1|| = 1$, $||a_2|| = 2$, and $||a_3|| = 3$. Find the Gramian $A^{\mathsf{T}}A$ of A.

Problem 5. Consider the matrices A, B, and Y given by

$$A = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & -1 \\ -5 & -4 & 0 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} -2 & -2 & 0 \\ 2 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

(a) Calculate AB and BA. Are either of these matrices familiar?

(b) Find a matrix X satisfying AX = Y.

Row Echelon Forms

Name: ______ NetID: _____

Problem 1. Consider the matrix A, which is in reduced row echelon form and the vector \boldsymbol{v} given by

$$A = \begin{bmatrix} 1 & 0 & 0 & -5 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \qquad \mathbf{v} = \begin{bmatrix} * \\ * \\ * \\ 1 \\ * \end{bmatrix}$$

Find the missing entries in v so that Av = 0.

Hint. Note that the columns of A satisfy $a_4 = -5 \cdot a_1 - 3 \cdot a_2 + 7 \cdot a_3$.

Problem 2. Consider the matrix A, which is in reduced row echelon form and the vectors v_1 and v_2 given by

$$A = \begin{bmatrix} 1 & 0 & 7 & 0 & -9 \\ 0 & 1 & 4 & 0 & -2 \\ 0 & 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \qquad \mathbf{v}_1 = \begin{bmatrix} * \\ * \\ 1 \\ * \\ 0 \end{bmatrix} \qquad \qquad \mathbf{v}_2 = \begin{bmatrix} * \\ * \\ 0 \\ * \\ 1 \end{bmatrix}$$

Find the missing entries in v_1 and v_2 so that $Av_1 = 0$ and $Av_2 = 0$.

Hint. The pivot columns of A are $\{a_1, a_2, a_4\}$ and the nonpivot columns of A are $\{a_3, a_5\}$. Start by expressing each nonpivot column as a linear combination of the pivot columns.

Problem 3. There is only one 4×6 matrix R in reduced row echelon form with column relations

 $\operatorname{Col}_2 = -2 \cdot \operatorname{Col}_1$ $\operatorname{Col}_5 = 2 \cdot \operatorname{Col}_1 + \operatorname{Col}_4$ $\operatorname{Col}_6 = \operatorname{Col}_1 - \operatorname{Col}_3 + 2 \cdot \operatorname{Col}_4$

(a) What is rank(R)?

(b) Find R.

Problem 4. Consider the matrix A, which is in row echelon form and the vector \boldsymbol{v} given by

$A = \begin{bmatrix} -1 & 1 & 1 & -1 & -1 \\ 0 & 6 & 2 & -4 & -3 \\ 0 & 0 & -4 & -2 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix} \qquad \qquad \mathbf{v} = \begin{bmatrix} \mathbf{v} \\ \mathbf{v} \\ \mathbf{v} \\ \mathbf{v} \\ \mathbf{v} \\ -\mathbf{e} \\ \mathbf{e} \end{bmatrix}$	$ \begin{array}{c} 1 \\ 2 \\ 3 \\ 6 \\ 0 \end{array} $
--	--

Find the missing entries in v so that Av = 0.

Problem 5. Suppose A is 8×13 and in row echelon form. Is it possible for rank(A) = nullity(A)? Why or why not?

Linear Systems

Name: _______ NetID: ______

Problem 1. Consider the reduced row echelon form system given by

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & -2 & 4 & 7 & 9 \end{bmatrix}$$

Find the general solution to this system.

Problem 2. Consider the reduced row echelon form system Ax = b where

$$A = \begin{bmatrix} 1 & 0 & -4 & 0 & -12 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -8 \\ 4 \\ 0 \end{bmatrix}$$

Find the general solution to this system.

Problem 3. Consider the reduced row echelon form system Rx = b where

Write the general solution \boldsymbol{x} to $R\boldsymbol{x} = \boldsymbol{b}$ as $\boldsymbol{x}_p + c_1 \cdot \boldsymbol{x}_1 + c_2 \cdot \boldsymbol{x}_2 + c_3 \cdot \boldsymbol{x}_3$. Calculate the matrix-vector products $R\boldsymbol{x}_1$, $R\boldsymbol{x}_2$, and $R\boldsymbol{x}_3$.

Problem 4. Consider the row echelon form system Ax = b where

$$A = \begin{bmatrix} 2 & -4 & 5 & 3 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 6 \end{bmatrix} \qquad \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \qquad \mathbf{b} = \begin{bmatrix} 4 \\ -2 \\ -12 \end{bmatrix}$$

Use back-substitution to find the general solution to this system.

Gauss-Jordan Elimination

Name: _______ NetID: ______

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 3 & -9 & 0 & 0 & 15 \\ -2 & 6 & 0 & 0 & -10 \\ 7 & -21 & 4 & 0 & 79 \\ 9 & -27 & 2 & 8 & 51 \end{bmatrix}$$

Calculate $\operatorname{rref}(A)$ by applying the Gauß-Jordan algorithm. You can (and should!) check your work on a computer, but you must precisely follow the algorithm (including labels on your row-reductions) and show each step here.

Problem 2. Let A be the incidence matrix of the digraph



Find $\operatorname{rref}(A)$ (on a computer, if you're confident you could do it by hand) and use this to calculate $\operatorname{nullity}(A)$ and $\operatorname{nullity}(A^{\intercal})$.

Problem 3. Let A be the incidence matrix of the digraph



Find $\operatorname{rref}(A)$ (on a computer, if you're confident you could do it by hand) and use this to calculate $\operatorname{nullity}(A)$ and $\operatorname{nullity}(A^{\intercal})$.

Problem 4. The matrix $A = \begin{bmatrix} 3 & 15 & -24 \\ -5 & -25 & 40 \\ 2 & 10 & -16 \\ -7 & -35 & 56 \end{bmatrix}$ is rank one. Find rref(A) without doing any row reductions.

Problem 5. Suppose that a matrix A satisfies

			A								
	-9	-43	*	-133	*		[1	0	5	0	1]
rref	3	16	*	48	*		0	1	-3	0	-2
	5	25	*	76	*	=	0	0	0	1	1
	-5	-24	*	-74	*		0	0	0	0	0

Note that two of the columns in A are currently unknown.

(a) Find the third column of A without doing any row reductions.

(b) Find the fifth column of A without doing any row reductions.

Name: ______ NetID: _____

Problem 1. Consider the matrix A, its inverse A^{-1} , and the vector **b** given by

$$A = \begin{bmatrix} -6 & -6 & 10 & 3\\ 2 & 3 & -5 & -3\\ 1 & 1 & -2 & -1\\ -2 & -2 & 3 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -1 & 3 & -3\\ 0 & 1 & -4 & -1\\ 1 & 0 & 0 & -3\\ -1 & 0 & -2 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} t\\ 4\\ 1\\ 0 \end{bmatrix}$$

Find all solutions \boldsymbol{x} to the system $A\boldsymbol{x} = \boldsymbol{b}$.

Problem 2. Consider the matrix L given by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

Find L^{-1} .

Problem 3. Suppose A is an $n \times n$ matrix that satisfies the equation

$$A^3 - 4A^2 + 7A - I_n = \boldsymbol{O}_n$$

Show that A is invertible and find a formula for A^{-1} .

Problem 4. Suppose A is a matrix satisfying

] [1]		[0]
4	-7		0
A	3	=	0
	2		0

Is this enough information to determine if A is nonsingular? Why or why not?

Problem 5. Supose A satisfies the equation

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is this enough information to determine if $(A^{\intercal}A)^{-1}$ exists? Why or why not?

EA = R Factorizations

Name: ______ NetID: _____

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 & 7 \\ -4 & -12 & 0 & 2 & -20 \\ 2 & 6 & 1 & -2 & -3 \end{bmatrix}$$

Use the Gauß-Jordan algorithm to find EA = R.

Problem 2. Consider the EA = R factorization

E					A						R					
[20	2	45	[0	[11]	-44	44	35	57	-59		[1	-4	4	0	2	1]
52	-14	109	-5	3	-12	12	10	16	-17		0	0	0	1	1	-2
2	1	5	0	-5	20	-20	-16	-26	27	=	0	0	0	0	0	0
-13	2	-28	1	-3	12	-12	-13	-19	23		0	0	0	0	0	0

(a) Suppose that **b** is a vector that makes the system $A\mathbf{x} = \mathbf{b}$ consistent. What system of equations must the coordinates of **b** solve and what is the coefficient matrix of this system?

(b) Let R' be the matrix obtained by deleting the last two rows of R. Find a matrix C satisfying A = CR' without doing any row reductions.

Name: ______ NetID: _____

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 2 & -3 & 5 & 0 & 7 \\ -4 & 6 & -10 & 4 & -7 \\ 6 & -9 & 13 & -11 & 24 \\ -2 & 3 & 3 & 36 & -28 \end{bmatrix}$$

Find a PA = LU factorization of A by applying the algorithm from class. You must show all of your steps.

Problem 2. Consider the PA = LU factorization and the vector **b** where

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & 3 & -1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -2 & 5 & 7 & 5 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ -3 \\ -1 \\ 1 \end{bmatrix}$$

Use this factorization and the method of back substitution to find the general solution to Ax = b.

Eigenvalues

Name: ______ NetID: _____

Problem 1. The following two matrices A and B have the same eigenvalues

	-3	1	0	0 -		-3	1	0	0]	
A =	0	-3	0	0	D	0	-3	1	0	
	0	0	7	1	D =	0	0	-3	0	
	0	0	0	7		0	0	0	7	

The lines in these matrices are provided purely for aesthetic purposes.

(a) What are the geometric multiplicities of the eigenvalues of A?

(b) What are the geometric multiplicities of the eigenvalues of B?

Problem 2. Consider the matrix A given by

$$A = \begin{bmatrix} -107 & 315 & -420 \\ -28 & 82 & -112 \\ 7 & -21 & 26 \end{bmatrix}$$

The eigenvalues of A are $\operatorname{E-Vals}(A) = \{5,-2\}\,.$

(a) Find all solutions to $(-2 \cdot I_3 - A)\mathbf{x} = \mathbf{0}$.

(b) Find all solutions to $(5 \cdot I_3 - A)\mathbf{x} = \mathbf{0}$.

Problem 3. Suppose that λ is an eigenvalue of an $n \times n$ matrix A. Show that λ is also an eigenvalue of A^{\intercal} . *Hint*. Recall that every matrix M satisfies $\operatorname{rank}(M^{\intercal}) = \operatorname{rank}(M)$.

Name: ______ NetID: _____

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} -8 & -12 & -60 & -48 \\ -1 & -2 & -9 & -7 \\ 3 & 5 & 24 & 19 \end{bmatrix}$$

(a) Let $\boldsymbol{v} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$. Show that $\boldsymbol{v} \notin \operatorname{Null}(A)$.

(b) Describe all vectors in Null(A).

Problem 2. Consider the matrix A given by

$$A = \begin{bmatrix} -1 & -3 & 0 & -6\\ -2 & -4 & 4 & -4\\ -1 & -3 & 1 & -5\\ 1 & 3 & -2 & 4 \end{bmatrix}$$

The scalar $\lambda = -1$ is an eigenvalue of A. Describe all of the vectors in the eigenspace $\mathcal{E}_A(-1)$.

Problem 3. Let A be an $n \times n$ matrix and define $M = A - 7 \cdot I_n$. Suppose that $\lambda = 3$ is an eigenvalue of A and that v is a corresponding eigenvector of A. Show that v is also an eigenvector of M and identify the corresponding eigenvalue.

Column Spaces

Name: ______ NetID: _____

Problem 1. Consider the matrix A and the vector v given by

Determine if $\boldsymbol{v} \in \operatorname{Col}(A)$.

Problem 2. Suppose EA = R where

Use this factorization to determine if $\boldsymbol{v} = \begin{bmatrix} 1 & 1 & 4 & 7 \end{bmatrix}^{\mathsf{T}}$ is in $\operatorname{Col}(A)$.

Problem 3. Consider the matrix N and the vector \boldsymbol{b} given by

$$N = \begin{bmatrix} -2 & 8 & -11 & -9 \\ 3 & -12 & 16 & 13 \end{bmatrix} \qquad \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

Suppose that A is a matrix with Col(A) = Null(N).

(a) How many rows does A have? Explain.

(b) Determine if the system $A\mathbf{x} = \mathbf{b}$ is consistent.

The Four Fundamental Subspaces

Name: ______ NetID: _____

Problem 1. Suppose that A is a $4 \times n$ matrix whose rows satisfy the equations

 $\operatorname{Row}_{1} - \operatorname{Row}_{2} + \operatorname{Row}_{3} + 2 \cdot \operatorname{Row}_{4} = \boldsymbol{O} \qquad \qquad 3 \cdot \operatorname{Row}_{1} - 2 \cdot \operatorname{Row}_{2} + \operatorname{Row}_{4} = \boldsymbol{O}$

Find two nonparallel vectors in $Null(A^{\intercal})$.

Problem 2. Let $V = \{ \begin{bmatrix} x & y \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^2 \mid xy = 0 \}$. Show that V is *not* a vector space. *Hint.* Note that $\begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathsf{T}} \in V$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathsf{T}} \in V$. Is every linear combination of these two vectors also in V? Problem 3. Consider the following conditions

 $\begin{bmatrix} 1 & -1 & 3 & 5 \end{bmatrix}^{\mathsf{T}} \in \operatorname{Col}(A^{\mathsf{T}}) \qquad \begin{bmatrix} 2 & 1 & 0 & 1 \end{bmatrix}^{\mathsf{T}} \in \operatorname{Col}(A^{\mathsf{T}}) \qquad \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}^{\mathsf{T}} \in \operatorname{Null}(A^{\mathsf{T}})$

Construct a matrix A with these properties and use a computer to find $\operatorname{rref}(A)$. *Hint.* Start by drawing the picture of the four fundamental subspaces and indicate where each of these three vectors belong in the picture.

Problem 4. Construct a matrix B such that $\text{Null}(B) = \text{Span}\{\begin{bmatrix} 1 & -4 & 5 & -6 \end{bmatrix}^{\intercal}, \begin{bmatrix} -7 & 29 & -30 & 39 \end{bmatrix}^{\intercal}\}$. Then, use a computer to find rref(B).

Hint. This is the same as asking for Col(A) = Null(B) for which A?

Linear Independence

Name: ______ NetID: _____

Problem 1. Suppose that $v_1, v_2, v_3 \in \mathbb{R}^n$ and let A be an $m \times n$ matrix.

(a) Suppose that $\{Av_1, Av_2, Av_3\}$ is linearly independent. Show that $\{v_1, v_2, v_3\}$ is linearly independent.

(b) Suppose that $\{v_1, v_2, v_3\}$ is linearly independent and that A has a left-inverse L (meaning $LA = I_n$). Show that $\{Av_1, Av_2, Av_3\}$ is linearly independent.
Problem 2. Let A be the incidence matrix of the digraph given by



(a) Are the columns of A linearly independent? Explain why or why not.

(b) Suppose we remove a_4 from the digraph, and let A' be the new incidence matrix. Are the columns of A' linearly independent? Explain why or why not.

(c) Suppose in part (b) we had removed a_3 instead of a_4 and let A' be the new incidence matrix. Are the columns of A' linearly independent? Explain why or why not.

Bases

Problem 1. Consider the matrices A and S given by

$$A = \begin{bmatrix} -40 & 50\\ -25 & 35 \end{bmatrix} \qquad \qquad S = \begin{bmatrix} 1 & -12\\ -12 & -6 \end{bmatrix}$$

These matrices have the same eigenvalues $\text{E-Vals}(A) = \text{E-Vals}(S) = \{-15, 10\}.$

(a) Find bases of $\mathcal{E}_A(-15)$ and $\mathcal{E}_A(10)$. Then take the inner-product of each basis vector of $\mathcal{E}_A(-15)$ and $\mathcal{E}_A(10)$. What is this inner-product?

(b) Find bases of $\mathcal{E}_S(-15)$ and $\mathcal{E}_S(10)$. Then take the inner-product of each basis vector of $\mathcal{E}_S(-15)$ and $\mathcal{E}_S(10)$. What is this inner-product?

Problem 2. Consider the data

$$A = \begin{bmatrix} 1 & 16 & -18 & -6 \\ -8 & 3 & -8 & -24 \\ -9 & 8 & -14 & -27 \\ 1 & -8 & 9 & 6 \end{bmatrix}$$
E-Vals(A) = {-5,3}

(a) Find a basis of the eigenspace $\mathcal{E}_A(-5)$.

(b) Find a basis of the eigenspace $\mathcal{E}_A(3)$.

(c) Count the number of basis vectors you found for $\mathcal{E}_A(-5)$ and $\mathcal{E}_A(3)$. How do these counts compare to the geometric multiplicities $\operatorname{gm}_A(-5)$ and $\operatorname{gm}_A(3)$?

Dimension

Name: ______ NetID: _____

Problem 1. If possible, construct a matrix A with the following properties

 $\begin{bmatrix} -5 & 12 & -41 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} -3 & 7 & -24 \end{bmatrix}^{\mathsf{T}} \in \operatorname{Null}(A)$ $\begin{bmatrix} 1 & 0 & 4 & 0 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} 0 & 1 & 5 & -7 \end{bmatrix}^{\mathsf{T}} \in \operatorname{Col}(A)$

If this is not possible, then explain why.

Problem 2. Let A be the incidence matrix of the digraph G given by



(a) Draw the picture of the four fundamental subspaces of A. Label each space with its dimension.

(b) Is it possible to add three nodes and nine arrows to G to construct a new digraph G' whose circuit rank is 13? Rigorously argue why or why not using concepts covered in class.

Problem 3. Consider
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 10 & -15 & 0 \\ 0 & 0 & 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$
. Do not multiply to find A !

(a) Draw the picture of the four fundamental subspaces of A and include their dimensions.

(b) Find the pivot columns of A. These vectors form a basis of _____.

(c) Find a basis of $\operatorname{Col}(A^{\intercal})$.

(d) Find a basis of Null(A).

(e) List every vector in $\text{Null}(A^{\intercal})$.

Orthogonality

Name: ______ NetID: _____

Problem 1. Let S be the vector space given by

 $S = \text{Span}\{\begin{bmatrix} 1 & 2 & -6 & -1 & -3 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} 1 & 3 & -8 & -2 & -4 \end{bmatrix}^{\mathsf{T}}\}$

Find a basis of S^{\perp} . Put your basis vectors into the rows of a matrix B and use a computer to find $\operatorname{rref}(B)$.

Problem 2. Let P be the vector space of solutions to the equations

$$-9x_1 + 47x_2 - 38x_3 - 112x_4 + 74x_5 = 0 \qquad -5x_1 + 26x_2 - 21x_3 - 62x_4 + 41x_5 = 0$$

Find a basis of P^{\perp} . Put your basis vectors into the rows of a matrix B and use a computer to find $\operatorname{rref}(B)$.

Problem 3. Let A be the incidence matrix of the digraph G given by



Let $\boldsymbol{b} = \begin{bmatrix} 5 & -3 & -1 & -1 \end{bmatrix}^{\mathsf{T}}$. Determine if the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent without finding A.

Hint. We can find a basis of Null(A^{\intercal}) without finding A. How would this basis help in determining if Ax = b is consistent?

Projections

Name: ______ NetID: _____

Problem 1. Suppose that an $n \times n$ matrix P is symmetric $(P^{\intercal} = P)$ and idempotent $(P^2 = P)$. (a) Show that the matrix $I_n - P$ is symmetric.

(b) Show that the matrix $I_n - P$ is idempotent.

(c) Show that every vector $\boldsymbol{v} \in \mathbb{R}^n$ satisfies the equation

$$\langle P\boldsymbol{v}, (I_n - P)\boldsymbol{v} \rangle = 0$$

Hint. Start with the adjoint formula.

Problem 2. Let $V \subset \mathbb{R}^5$ be the vector space of solutions to the equation

$$x_1 - x_2 - 3x_3 - 2x_4 + 2x_5 = 0$$

(a) Find a matrix A such that Null(A) = V. Draw the picture of all four fundamental subspaces of A (including their dimensions).

(b) Find the projection matrix P_V without using a computer. Hint. This is a difficult direct computation. The picture from part (a) helps make this problem easier!

Least Squares

Name: ______ NetID: _____

Problem 1. Consider the data points

 $\{(0,0), (1,8), (3,8), (4,20)\}$

Find the parabola of best fit $\widehat{f}(t) = \widehat{a}_0 + \widehat{a}_1 t + \widehat{a}_2 t^2$ to this data.

Problem 2. Consider the weighted digraph given by



Let A be the incidence matrix and let $\boldsymbol{b} = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \end{bmatrix}^{\mathsf{T}}$.

(a) Show that $A\boldsymbol{w} = \boldsymbol{b}$ is inconsistent without doing any row-reductions or any computer calculations. *Hint.* The consistency of $A\boldsymbol{w} = \boldsymbol{b}$ hinges on whether or not $\boldsymbol{b} \in \operatorname{Col}(A) = \operatorname{Null}(A^{\intercal})^{\perp}$. Start by finding a basis of $\operatorname{Null}(A^{\intercal})$.

(b) Find the least squares approximate solution \hat{w} to Aw = b. Here, you may use a computer but you must explain your reasoning.

A = QR Factorizations

Name: ______ NetID: _____

Problem 1. Suppose that A is a 2021×3 matrix whose columns are mutually orthogonal and have lengths 10, 11, and 12 respectively. Find $A^{\intercal}A$.

Problem 2. Suppose that Q_1 and Q_2 have orthonormal columns. Show that $Q = Q_1 Q_2$ also has orthonormal columns.

Problem 3. Consider the data

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -4 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -1/6 \\ 1/2 & -5/6 \\ -1/2 & -1/6 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 3 \end{bmatrix} \qquad \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ -6 \\ 4 \end{bmatrix}$$

(a) Use the QR-factorization to find the least-squares approximate solution \hat{x} to Ax = b.

(b) Use the QR-factorization to find the projection of \boldsymbol{b} onto $\operatorname{Col}(A)$.

(c) Suppose that $\{q_1, q_2, q_3, q_4\}$ is an orthonormal basis of \mathbb{R}^4 where $\{q_1, q_2\}$ are the two columns of Q. Which (if any) of the four fundamental subspaces of A do $\{q_3, q_4\}$ belong to? Explain.

The Gram-Schmidt Algorithm

Name: ______ NetID: _____

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 2 & 4 & -6 \\ -2 & 0 & -1 \\ 0 & -3 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

Use the Gram-Schmidt algorithm to find a QR factorization of A. You can do calculations on a computer, but you must show every step of the algorithm to receive credit.

Problem 2. The Gram-Schmidt algorithm can be used as an alternative to row-reducing to determine the pivot and nonpivot columns of a matrix. Consider the matrix A given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 3\\ 0 & 1 & -1 & 3\\ -1 & -3 & 6 & -5\\ 1 & 3 & -6 & 3 \end{bmatrix}$$

Find the vectors $\{w_1, w_2, w_3, w_4\}$ according to the Gram-Schmidt algorithm. If any w_k is the zero vector, then do not use w_k to calculate the subsequent vectors. The kth column of A is a pivot column if and only if $w_k \neq 0$!

Use this procedure to determine the pivot and nonpivot columns of A.

Name: ______ NetID: _____

Problem 1. Recall that a matrix A is called *idempotent* if $A^2 = A$. Show that every idempotent matrix A satisfies det(A) = 1 or det(A) = 0.

Problem 2. Consider two matrices A and B of the form

$$A = \begin{bmatrix} \operatorname{Row}_1 \\ \operatorname{Row}_2 \\ \operatorname{Row}_3 \end{bmatrix} \qquad \qquad B = \begin{bmatrix} \operatorname{Row}_1 + \operatorname{Row}_2 + \operatorname{Row}_3 \\ \operatorname{Row}_1 + \operatorname{Row}_2 \\ \operatorname{Row}_1 \end{bmatrix}$$

Suppose det(A) = -11. Find det(B).

Problem 3. Suppose that A is 5×5 with det(A) = 3. Find $det(-2 \cdot A)$. In general, what is $det(c \cdot A)$ if A is $n \times n$?

Problem 4. A matrix S is called *skew-symmetric* if $S^{\intercal} = -S$. Suppose that S is 2023×2023 and skew-symmetric. Find det(S).

Name: ______ NetID: _____

Problem 1. Consider the matrix X given by

$$X = \begin{bmatrix} t-5 & -2 & -4 \\ 8 & t+2 & 10 \\ 4 & 1 & t+5 \end{bmatrix}$$

Find all values of t for which X is singular.

Problem 2. Consider the system of equations given by

Use Cramer's rule to solve this system.

The cross product of $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ is the "symbolic determinant" given by

$$m{v} imes m{w} = egin{bmatrix} m{e}_1 & m{e}_2 & m{e}_3 \ v_1 & v_2 & v_3 \ w_1 & w_2 & w_3 \end{bmatrix} = egin{bmatrix} m{v}_2 & v_3 \ w_2 & w_3 \end{bmatrix} - egin{bmatrix} v_1 & v_3 \ w_1 & w_3 \end{bmatrix} egin{bmatrix} v_1 & v_2 \ w_1 & w_2 \end{bmatrix}^{\mathsf{T}}$$

Note that $\boldsymbol{v} \times \boldsymbol{w}$ is a vector.

Problem 3. Let $\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w}$ where $\boldsymbol{v} = \begin{bmatrix} 3 & -7 & 5 \end{bmatrix}^{\mathsf{T}}$ and $\boldsymbol{w} = \begin{bmatrix} 5 & 4 & 2 \end{bmatrix}^{\mathsf{T}}$. Find \boldsymbol{n} and calculate the inner products $\langle \boldsymbol{n}, \boldsymbol{v} \rangle$ and $\langle \boldsymbol{n}, \boldsymbol{w} \rangle$.

Problem 4. The *scalar triple product* states that any three vectors $v, w, x \in \mathbb{R}^3$ satisfy the equation

$$\langle \boldsymbol{x}, \boldsymbol{v} imes \boldsymbol{w}
angle = egin{bmatrix} x_1 & x_2 & x_3 \ v_1 & v_2 & v_3 \ w_1 & w_2 & w_3 \end{bmatrix}$$

Use this equation to show that v is orthogonal to $v \times w$.

Hint. What is the rank of the matrix in this equation when x = v?

Problem 5. Consider the parallelogram \mathcal{P} formed by two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$.



The area of this parallelogram is given by $\operatorname{area}(\mathcal{P}) = \| \boldsymbol{v} \times \boldsymbol{w} \|$. Use this fact to calculate the area of the following parallelogram



Complex Numbers

Name: ______ NetID: _____

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 1 & i & i \\ i & -1 & i \end{bmatrix}$$

(a) Use elementary row operations to calculate rref(A) without a computer.

(b) Find a unit vector \boldsymbol{u} in Null(A).

Problem 2. Suppose that $A^* = -A$. Show that H = iA is Hermitian.

Problem 3. Suppose that $\boldsymbol{u} \in \mathbb{C}^n$ is a unit vector and define $Q = I_n - 2\boldsymbol{u}\boldsymbol{u}^*$. (a) Show that Q is *Hermitian*.

(b) Show that Q is unitary.

Polynomial Algebra

Name: ______ NetID: _____

Problem 1. Let r_1 , r_2 , and r_3 be the three (not necessarily distinct) roots of the cubic

 $f(t) = t^3 + c_2 t^2 + c_1 t + c_0$

Prove that $c_1 = r_1 \cdot r_2 + r_1 \cdot r_3 + r_2 \cdot r_3$.

Problem 2. Consider the polynomial f(t) given by

$$f(t) = -67t^9 - 3t^8 + 4t^7 - 9t^6 + t^5 - t^4 - 2t^3 + t^2 + t + 4t^6 + t^6 + t^6 - t^6 + t^6 - t^6 + t^6$$

This polynomial has nine distinct roots. Find the sum and product of these roots.

The Characteristic Polynomial

Name: ______ NetID: _____

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & y & z \end{bmatrix}$$

Since A is 3×3 , the characteristic polynomial of A is of the form $\chi_A(t) = t^3 + c_2 t^2 + c_1 t + c_0$. (a) Find c_0, c_1 , and c_2 .

(b) Suppose that E-Vals $(A) = \{-3, 5\}$ with $\operatorname{am}_A(-3) = 1$ and $\operatorname{am}_A(5) = 2$. Find x, y, and z.

Problem 2. Consider the matrix A given by

$$A = \begin{bmatrix} -25 & 36\\ -18 & 29 \end{bmatrix}$$

Find all eigenvalues and bases for all eigenspaces of A without using a computer.

Name: ______ NetID: _____

Problem 1. Suppose that A is similar to B. Show that $\chi_A(t) = \chi_B(t)$.

Problem 2. Suppose that $A \sim B$ so $A = XBX^{-1}$ and consider $\boldsymbol{v} \in \mathcal{E}_B(\lambda)$. Show that $X\boldsymbol{v} \in \mathcal{E}_A(\lambda)$.

Problem 3. There is only one matrix A whose eigenspaces are

$$\mathcal{E}_A(-3) = \operatorname{Span}\left\{ \begin{bmatrix} 1\\0\\0\\1 \end{bmatrix} \right\} \qquad \qquad \mathcal{E}_A(5) = \operatorname{Span}\left\{ \begin{bmatrix} 0\\1\\-1\\0 \end{bmatrix} \right\} \qquad \qquad \mathcal{E}_A(11) = \operatorname{Span}\left\{ \begin{bmatrix} -2\\1\\0\\-2 \end{bmatrix}, \begin{bmatrix} -1\\1\\1\\0 \end{bmatrix} \right\}$$

Find A. You may do any calculations on a computer, but you must show your steps and explain your reasoning.

Problem 4. Consider the matrix A given by

$$A = \begin{bmatrix} -47 & 12 & -204 & -24 \\ 20 & -7 & 88 & 4 \\ 12 & -4 & 53 & 4 \\ -4 & 4 & -20 & 5 \end{bmatrix}$$

Show that A is diagonalizable and find X and D so that $A = XDX^{-1}$. You are free to outsource annoying calculations to a computer (like factoring $\chi_A(t)$) but you must show your steps to receive credit.

This page is intentionally left blank and should be used to continue your solution if you need extra space. You must upload this page to Gradescope, even if you leave this page blank.

Matrix Exponentials

Name: ______ NetID: _____

Problem 1. Suppose that A is idempotent. Prove that $\exp(At) = I + (e^t - 1)A$.

Problem 2. Calculate $\exp(At)$ for $A = \begin{bmatrix} -26 & -39 \\ 18 & 27 \end{bmatrix}$ by using the result from the previous problem.

Problem 3. Suppose that P is projection in \mathbb{R}^2 onto the line y = x. Calculate $\lim_{t \to \infty} u(t)$ where u(t) is the solution u(t) to the initial value problem u' = -Pu with $u(0) = \begin{bmatrix} 3 & 1 \end{bmatrix}^{\mathsf{T}}$.
Problem 4. A rabbit population r shows fast growth but is vulnerable to a wolf population w. This predator-prey model is exhibited by the initial value problem

r' = 6r - 2w w' = 2r + w r(0) = 30 w(0) = 30

What are the populations at time t? After a long time, what is the ratio of rabbits to wolves?

The Spectral Theorem

Name: ______ NetID: _____

Problem 1. Consider the real-symmetric matrix S given by

$$S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

Find all unitary matrices U that diagonalize S.

Problem 2. Consider the real-symmetric matrix S given by

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

Find a spectral factorization of S without using a computer.

Problem 3. A matrix S is skew-Hermitian if $S^* = -S$. Suppose that λ is an eigenvalue of a skew-Hermitian matrix S. Show that $\lambda = -\overline{\lambda}$.

Hint. Follow the proof of Part I of the spectral theorem.

Problem 4. The real symmetric matrix S below has exactly three eigenvalues.

$$S = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 4 & 5 & 4 & 0 \\ -3 & 4 & 4 & -1 \\ 1 & 0 & -1 & 8 \end{bmatrix} \quad \mathcal{E}_S(\lambda_1) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \mathcal{E}_S(\lambda_2) = \operatorname{Span}\{\boldsymbol{u}_2\} \quad \mathcal{E}_S(9) = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right\}$$

(a) Find the correct value of λ_1 .

(b) Find the correct value of λ_2 .

(c) There are only two valid unit vectors that u_2 could equal. Find these two unit vectors. You may need to do some row reductions on a computer.

(d) Find a spectral factorization $S = UDU^{\intercal}$.

Definiteness

Name: ______ NetID: _____

Problem 1. Let $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$ where

 $S = \begin{bmatrix} 10 & 3\\ 3 & 2 \end{bmatrix}$

Complete the square to write $q(\mathbf{x}) = \lambda_1 y_1^2 + \lambda_2 y_2^2$ and determine the definiteness of S. Write each of y_1 and y_2 in terms of x_1 and x_2 .

Problem 2. Suppose that S_1 and S_2 are positive definite matrices. Show that $S_1 + S_2$ is also positive definite.

Hint. Consider $q(\boldsymbol{x}) = \langle \boldsymbol{x}, (S_1 + S_2) \boldsymbol{x} \rangle$.

Problem 3. Suppose d_1 , d_2 , and d_3 are real numbers and consider the matrix S given by

		L			D			L^{\intercal}	
	[1	0	0	d_1	0	[0	[1	-3	7]
S =	-3	1	0	0	d_2	0	0	1	-9
	7	-9	1	0	0	d_3	0	0	1

Let $q(\boldsymbol{x})$ be the quadratic form $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$.

(a) Prove that S is real-symmetric without calculating S.

(b) It is possible to write $q(x) = d_1 y_1^2 + d_2 y_2^2 + d_3 y_3^2$. Find y_1, y_2 , and y_3 in terms of x_1, x_2 , and x_3 .

(c) Under what condition is S positive definite? Clearly explain your condition.

Problem 4. Consider the quadratic form on \mathbb{R}^4 given by

$$q(\mathbf{x}) = \frac{2}{36}(5x_1 - 3x_2 - x_3 + x_4)^2 + \frac{1}{6}(x_1 + x_2 + 2x_3)^2 + \frac{-2}{12}(x_1 + x_2 - x_3 - 3x_4)^2 + \frac{-7}{18}(x_1 + 3x_2 - 2x_3 + 2x_4)^2 + \frac{-7}{18}(x_1 + 2x_2 - 2x_4)^$$

Let S be the real-symmetric matrix satisfying $q(x) = \langle x, Sx \rangle$.

(a) Find a spectral factorization $S = UDU^{\dagger}$ of S without finding S and determine the definiteness of S.

(b) Show that $\exp(S)$ is real symmetric and determine the definiteness of $\exp(S)$.

Singular Value Decomposition

Name:	NetID:

Problem 1. Find a singular value decomposition of $A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \\ -1 & 0 \end{bmatrix}$ without using a computer.

Problem 2. Use your work from Problem 1 to find the rank one approximation of the matrix A from Problem 1 (you are graded on correctness, so make sure your work in Problem 1 is correct).

Problem 3. Suppose that $A^*A = VDV^*$ where D is real-diagonal and invertible and V has orthonormal columns. Further suppose that $\Sigma = \sqrt{D}$. Show that $U = AV\Sigma^{-1}$ has orthonormal columns.

Problem 4. Suppose that A is a matrix satisfying

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\2\end{bmatrix} \qquad \qquad A\begin{bmatrix}2\\-2\end{bmatrix} = \begin{bmatrix}2\\1\\-2\end{bmatrix}$$

Find a singular value decomposition of A.

Problem 5. Suppose that A = QR.

(a) Show that A and R have the same singular values.

(b) Suppose that $R = U_R \Sigma_R V_R^*$ is a singular value decomposition. Find a singular decomposition $A = U \Sigma V^*$.

Problem 6. Find the rank one approximation of $A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & -2 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

Linear Approximations

Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{f}(x,y) = \begin{bmatrix} x^2 + \sin(x) \\ x(y-2) \\ -3xy + y^2 \end{bmatrix}$$

Use the local linearization of \boldsymbol{f} at the point P = (0, 1) to approximate f(1/2, 1/2).

Problem 2. Let $r : \mathbb{R} \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{r}(t) = \begin{bmatrix} \cos\left(t\right) \\ \sin\left(t\right) \\ t \end{bmatrix}$$

Use the local linearization of \boldsymbol{r} at $t = 1/2 \pi$ to approximate $\boldsymbol{r}(1/2 \pi + 1/3)$.

Problem 3. Consider $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{F}(r,\theta,z) = \begin{bmatrix} r\cos\left(\theta\right) \\ r\sin\left(\theta\right) \\ z \end{bmatrix}$$

Show that $\det(D\mathbf{F}) = r$.

Problem 4. Consider $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{F}(\rho,\varphi,\theta) = \begin{bmatrix} \rho\cos\left(\theta\right)\sin\left(\varphi\right) \\ \rho\sin\left(\theta\right)\sin\left(\varphi\right) \\ \rho\cos\left(\varphi\right) \end{bmatrix}$$

Show that $\det(D\mathbf{F}) = \rho^2 \sin(\varphi)$.

Cholesky Factorizations

	4	-2	-2	-20]	
D \mathbf{D} D	-2	1	1	10	
Problem 1. Find the definiteness of $S =$	-2	1	17	14	·
	-20	10	14	105	
	-			-	

Problem 2. Calculate $S = LDL^{\intercal}$ where $S = \begin{bmatrix} 2 & 12 & 10 \\ 12 & 67 & 105 \\ 10 & 105 & -351 \end{bmatrix}$.

Problem 3. Suppose that factoring $S = LDL^{\intercal}$ allows us to write the quadratic form $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$ as

$$q(\mathbf{x}) = 10 (x_1 - 5 x_2 + 2 x_3)^2 - 11 (x_2 - 6 x_3)^2 - 5 x_3^2$$

Find L and D and determine the definitess of S.

Problem 4. Determine the definiteness of $S = \begin{bmatrix} 0 & 4 & -6 & 8 & 16 \\ 4 & -650 & 3 & 1941 & -1 \\ -6 & 3 & 8 & -144 & 16 \\ 8 & 1941 & -144 & 2 & 18 \\ 16 & -1 & 16 & 18 & 52 \end{bmatrix}$.

Problem 5.	Find R the Cholesky factorization $S = R^{\intercal}R$ of $S =$	$9 \\ 15 \\ -6$	$15 \\ 29 \\ 8$	$\begin{bmatrix} -6 \\ 8 \\ 206 \end{bmatrix}$
		-6	8	206

The Hessian

Problem 1. Consider the point P(17, -22, 37) and suppose $f \in \mathscr{C}(\mathbb{R}^3)$ is a scalar field satisfying

$$Hf(P) = \begin{bmatrix} -4 & 2 & -2\\ 2 & -2 & 3\\ -2 & 3 & -8 \end{bmatrix}$$

Suppose we use the local linearization of f at P to estimate f(17 - 1, -22 + 1, 37 + 1). Do we expect this estimation to be an overestimate or an underestimate? Explain.

Problem 2. Consider the point P(1,1,1) and the scalar field $f \in \mathscr{C}(\mathbb{R}^3)$ given by

$$f(x, y, z) = x^3 z + y^3 + x^2 z + z^3$$

(a) Use the local linearization of f at P to approximate $f(1 + \frac{1}{4}, 1 - \frac{1}{3}, 1 - \frac{1}{5})$.

(b) Use the second degree Taylor polynomial of f at P to approximate $f(1 + \frac{1}{4}, 1 - \frac{1}{3}, 1 - \frac{1}{5})$.

(c) Do you expect your linear approximation in part (a) to be an overestimate or undestimate?