Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: Solutions

NetID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

February 4, 2022

- There are 100 points and 8 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

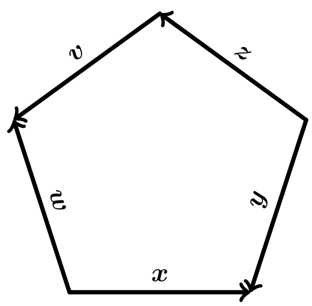
(4 pts) **Problem 1.** Suppose that A is a 5×4 matrix. Fill in the entries in the vector $\boldsymbol{v}$ below to make the equation true.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \overset{\boldsymbol{v}}{\begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}} = \text{(twice the second column of } A\text{) minus (the third column of } A\text{)}$$

(4 pts) **Problem 2.** One of the following vectors cannot be expressed as a linear combination of $\begin{bmatrix}3 & 0 & -5\end{bmatrix}^\intercal$ and $\begin{bmatrix}7 & 0 & 9\end{bmatrix}^\intercal$. Select this vector.

○ $\begin{bmatrix}0 & 0 & 0\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}46 & 0 & 6\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}18 & 0 & 32\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}10 & 0 & 4\end{bmatrix}^\intercal$ √ $\begin{bmatrix}0 & 1 & 0\end{bmatrix}^\intercal$

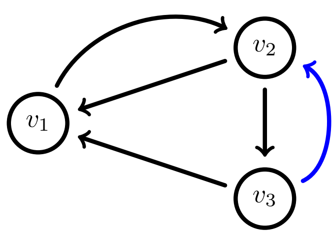
(4 pts) **Problem 3.** Suppose that $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ are vectors represented geometrically by arrows that fit into the diagram below.



$$\boldsymbol{v} = \underline{\quad 1 \quad} \cdot \boldsymbol{w} + \underline{\quad -1 \quad} \cdot \boldsymbol{x} + \underline{\quad 1 \quad} \cdot \boldsymbol{y} + \underline{\quad -1 \quad} \cdot \boldsymbol{z}$$

Fill in the blanks in the above equation to correctly express $\boldsymbol{v}$ as a linear combination of $\boldsymbol{w}$, $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

Problem 4. Consider the digraph G and its incidence matrix A depicted below.



$$A = \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 0 & 0 \end{bmatrix}$$

Note that G is missing an arrow!

(4 pts) (a) Draw the missing arrow in the figure depicting G above.

(6 pts) (b) The incidence matrix A satisfies $\operatorname{nullity}(A^\intercal) = 1$. Use this information to fill in the blanks below.

$\operatorname{rank}(A) = \underline{\quad 2 \quad}$ $\qquad$ $\operatorname{nullity}(A) = \underline{\quad 3 \quad}$ $\qquad$ $\operatorname{rank}(A^\intercal) = \underline{\quad 2 \quad}$

(8 pts) **Problem 5.** Suppose that $S = A + A^\intercal$ where A is 2022×2022. Show that S is symmetric.

Solution. $S^\intercal = (A + A^\intercal)^\intercal = A^\intercal + (A^\intercal)^\intercal = A^\intercal + A = S$

(20 pts) **Problem 6.** Consider the matrix A given by

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ 1 & 6 & 5 & 0 \\ -1 & -1 & 5 & 1 \end{bmatrix}$$

Use the Gauß-Jordan algorithm to calculate $\operatorname{rref}(A)$. You must follow the algorithm precisely and correctly label each row-reduction to receive credit.

Solution. Following the algorithm, we have

$$\overset{A}{\begin{bmatrix} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ 1 & 6 & 5 & 0 \\ -1 & -1 & 5 & 1 \end{bmatrix}} \xrightarrow{\substack{\boldsymbol{r}_2 + 2\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_2 \\ \boldsymbol{r}_3 - \boldsymbol{r}_1 \to \boldsymbol{r}_3 \\ \boldsymbol{r}_4 + \boldsymbol{r}_1 \to \boldsymbol{r}_4}} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{\boldsymbol{r}_2 \leftrightarrow \boldsymbol{r}_3} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{1/4\cdot\boldsymbol{r}_2 \to \boldsymbol{r}_2} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{\boldsymbol{r}_1 - 2\cdot\boldsymbol{r}_2 \to \boldsymbol{r}_1 \\ \boldsymbol{r}_4 - \boldsymbol{r}_2 \to \boldsymbol{r}_4}} \begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\boldsymbol{r}_3 \leftrightarrow \boldsymbol{r}_4} \overset{\operatorname{rref}(A)}{\begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Problem 7. Suppose that A is a nonsingular matrix that satisfies the following equation.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \overset{B}{\begin{bmatrix} -1 & 0 & -4 & 1 & -1 \\ -5 & 0 & -5 & 4 & -1 \\ -1 & 1 & 1 & -5 & 1 \\ 4 & 0 & 2 & 2 & 0 \end{bmatrix}} = \begin{bmatrix} -4 & 0 & -1 & 3 & 0 \\ 2 & 1 & -2 & -7 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 3 & 1 & 1 & -13 & 1 \end{bmatrix}$$

(4 pts) (*a*) The number of columns of A is __4__ and the number of rows of A is __4__.

(7 pts) (*b*) Find the third column of A. Clearly explain your reasoning to receive credit.

Solution. The second column of B is $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\intercal$. Since the third column of A is $A\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\intercal$, it follows that the third column of A is the second column of AB, which is $\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^\intercal$.

(7 pts) (*c*) Let $\boldsymbol{b} = \begin{bmatrix} -1 & -2 & 1 & 1 \end{bmatrix}^\intercal$ (the third column of AB). Find all solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your reasoning to receive credit.

Solution. Note that $\boldsymbol{b} = \begin{bmatrix} -1 & -2 & 1 & 1 \end{bmatrix}^\intercal$ is the third column of AB, which means chosing $\boldsymbol{x} = \begin{bmatrix} -4 & -5 & 1 & 2 \end{bmatrix}^\intercal$ (the third column of B) gives a solution to $A\boldsymbol{x} = \boldsymbol{b}$. There are no other solutions because A is nonsingular.

(7 pts) (*d*) Find the last column of A^{-1}. Clearly explain your reasoning to receive credit.

Hint. Start by explaining what the last column of AA^{-1} is.

Solution. Recall that $AA^{-1} = I_4$. This means that the last column of AA^{-1} is $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\intercal$, which is also the last column of AB. Hence the last column of B is the last column of A^{-1}.

Problem 8. Consider the $EA = R$ factorization given by

$$\overset{E}{\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -3 \\ 1 & 0 & 2 & -1 \\ 0 & -1 & 0 & 3 \end{bmatrix}} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \overset{R}{\begin{bmatrix} 1 & 0 & 3 & 0 & -2 \\ 0 & 1 & -6 & 0 & -5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

(5 pts) (a) Which of the columns of A is a *nonpivot column*? Select all that apply.

○ Col_1 ○ Col_2 √ Col_3 ○ Col_4 √ Col_5

(10 pts) (b) Suppose that the first column of A is $\begin{bmatrix} -1 & -3 & 0 & -1 \end{bmatrix}^\intercal$ and that the second column of A is $\begin{bmatrix} 2 & 0 & -1 & 0 \end{bmatrix}^\intercal$. Find the third column of A. Clearly explain your reasoning to receive credit.

Solution. The third column of A is a nonpivot column whose column relation is

$$\text{Col}_3 = 3\,\text{Col}_1 - 6\,\text{Col}_2 = 3 \cdot \begin{bmatrix} -1 \\ -3 \\ 0 \\ -1 \end{bmatrix} - 6 \cdot \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -15 \\ -9 \\ 6 \\ -3 \end{bmatrix}$$

(10 pts) (c) Let $\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^\intercal$. Find the full solution to $A\boldsymbol{x} = \boldsymbol{b}$ and write your solution in the form $\boldsymbol{x} = \boldsymbol{x}_p + c_1 \cdot \boldsymbol{x}_1 + \cdots + c_k \cdot \boldsymbol{x}_k$. If no solution exists, then explain why. Clearly explain your reasoning to receive credit.

Solution. Multiplying $A\boldsymbol{x} = \boldsymbol{b}$ on the left by E gives $EA\boldsymbol{x} = E\boldsymbol{b}$, which is $R\boldsymbol{x} = E\boldsymbol{b}$. In augmented form, this means that $[A \mid \boldsymbol{b}]$ reduces to $[R \mid E\boldsymbol{b}]$, which is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & -2 & 3 \\ 0 & 1 & -6 & 0 & -5 & -1 \\ 0 & 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Our dependent variables are $\{x_1, x_2, x_4\}$ and our free variables are $x_3 = c_1$ and $x_5 = c_2$. Solving for the dependent variables in terms of the free variables gives

$$\overset{\boldsymbol{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}} = \begin{bmatrix} -3\,c_1 + 2\,c_2 + 3 \\ 6\,c_1 + 5\,c_2 - 1 \\ c_1 \\ -3\,c_2 + 3 \\ c_2 \end{bmatrix} = \overset{\boldsymbol{x}_p}{\begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ 0 \end{bmatrix}} + c_1 \cdot \overset{\boldsymbol{x}_1}{\begin{bmatrix} -3 \\ 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + c_2 \cdot \overset{\boldsymbol{x}_2}{\begin{bmatrix} 2 \\ 5 \\ 0 \\ -3 \\ 1 \end{bmatrix}}$$