Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: ____________________ *NetID:* ____________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ____________________

February 4, 2022

- There are 100 points and 8 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

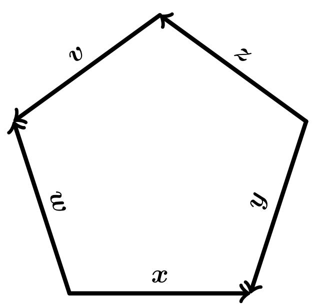
(4 pts) **Problem 1.** Suppose that A is a 5×4 matrix. Fill in the entries in the vector $\boldsymbol{v}$ below to make the equation true.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\overset{\boldsymbol{v}}{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}} = \text{(twice the second column of } A\text{) minus (the third column of } A\text{)}$$

(4 pts) **Problem 2.** One of the following vectors cannot be expressed as a linear combination of $\begin{bmatrix}3 & 0 & -5\end{bmatrix}^\intercal$ and $\begin{bmatrix}7 & 0 & 9\end{bmatrix}^\intercal$. Select this vector.

○ $\begin{bmatrix}0 & 0 & 0\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}46 & 0 & 6\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}18 & 0 & 32\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}10 & 0 & 4\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}0 & 1 & 0\end{bmatrix}^\intercal$

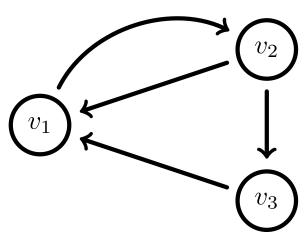
(4 pts) **Problem 3.** Suppose that $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ are vectors represented geometrically by arrows that fit into the diagram below.



$$\boldsymbol{v} = \underline{\qquad\quad} \cdot \boldsymbol{w} + \underline{\qquad\quad} \cdot \boldsymbol{x} + \underline{\qquad\quad} \cdot \boldsymbol{y} + \underline{\qquad\quad} \cdot \boldsymbol{z}$$

Fill in the blanks in the above equation to correctly express $\boldsymbol{v}$ as a linear combination of $\boldsymbol{w}$, $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

Problem 4. Consider the digraph G and its incidence matrix A depicted below.



$$A = \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 0 & 0 \end{bmatrix}$$

Note that G is missing an arrow!

(4 pts) (*a*) Draw the missing arrow in the figure depicting G above.

(6 pts) (*b*) The incidence matrix A satisfies $\operatorname{nullity}(A^\intercal) = 1$. Use this information to fill in the blanks below.

$$\operatorname{rank}(A) = \underline{\qquad\quad} \qquad \operatorname{nullity}(A) = \underline{\qquad\quad} \qquad \operatorname{rank}(A^\intercal) = \underline{\qquad\quad}$$

(8 pts) **Problem 5.** Suppose that $S = A + A^\intercal$ where A is 2022×2022. Show that S is symmetric.

(20 pts) **Problem 6.** Consider the matrix A given by

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ 1 & 6 & 5 & 0 \\ -1 & -1 & 5 & 1 \end{bmatrix}$$

Use the Gauß-Jordan algorithm to calculate $\operatorname{rref}(A)$. You must follow the algorithm precisely and correctly label each row-reduction to receive credit.

Problem 7. Suppose that A is a nonsingular matrix that satisfies the following equation.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \overset{B}{\begin{bmatrix} -1 & 0 & -4 & 1 & -1 \\ -5 & 0 & -5 & 4 & -1 \\ -1 & 1 & 1 & -5 & 1 \\ 4 & 0 & 2 & 2 & 0 \end{bmatrix}} = \begin{bmatrix} -4 & 0 & -1 & 3 & 0 \\ 2 & 1 & -2 & -7 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 3 & 1 & 1 & -13 & 1 \end{bmatrix}$$

(4 pts) (a) The number of columns of A is ________ and the number of rows of A is ________.

(7 pts) (b) Find the third column of A. Clearly explain your reasoning to receive credit.

(7 pts) (c) Let $\boldsymbol{b} = \begin{bmatrix} -1 & -2 & 1 & 1 \end{bmatrix}^{\intercal}$ (the third column of AB). Find all solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your reasoning to receive credit.

(7 pts) (d) Find the last column of A^{-1}. Clearly explain your reasoning to receive credit.
Hint. Start by explaining what the last column of AA^{-1} is.

Problem 8. Consider the $EA = R$ factorization given by

$$\overset{E}{\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -3 \\ 1 & 0 & 2 & -1 \\ 0 & -1 & 0 & 3 \end{bmatrix}} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \overset{R}{\begin{bmatrix} 1 & 0 & 3 & 0 & -2 \\ 0 & 1 & -6 & 0 & -5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

(5 pts) (a) Which of the columns of A is a *nonpivot column*? Select all that apply.

○ Col_1 ○ Col_2 ○ Col_3 ○ Col_4 ○ Col_5

(10 pts) (b) Suppose that the first column of A is $\begin{bmatrix} -1 & -3 & 0 & -1 \end{bmatrix}^\intercal$ and that the second column of A is $\begin{bmatrix} 2 & 0 & -1 & 0 \end{bmatrix}^\intercal$. Find the third column of A. Clearly explain your reasoning to receive credit.

(10 pts) (c) Let $\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^\intercal$. Find the full solution to $A\boldsymbol{x} = \boldsymbol{b}$ and write your solution in the form $\boldsymbol{x} = \boldsymbol{x}_p + c_1 \cdot \boldsymbol{x}_1 + \cdots + c_k \cdot \boldsymbol{x}_k$. If no solution exists, then explain why. Clearly explain your reasoning to receive credit.