Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: Solutions

NetID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

March 4, 2022

- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. The following row-reductions calculate the matrix U in $PA = LU$.

$$\overset{A}{\begin{bmatrix} 6 & -36 & 7 \\ 12 & -72 & 14 \\ -18 & 108 & -16 \\ -6 & 36 & 13 \end{bmatrix}} \xrightarrow{\substack{\boldsymbol{r}_2 - 2\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_2 \\ \boldsymbol{r}_3 + 3\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_3 \\ \boldsymbol{r}_4 + \boldsymbol{r}_1 \to \boldsymbol{r}_4}} \begin{bmatrix} 6 & -36 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 20 \end{bmatrix} \xrightarrow{\boldsymbol{r}_2 \leftrightarrow \boldsymbol{r}_3} \begin{bmatrix} 6 & -36 & 7 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 20 \end{bmatrix} \xrightarrow{\boldsymbol{r}_4 - 4\cdot\boldsymbol{r}_2 \to \boldsymbol{r}_4} \overset{U}{\begin{bmatrix} 6 & -36 & 7 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

(7 pts) (a) Find L.

Solution. Here we follow the algorithm from class.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 \end{bmatrix} \to \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{bmatrix}}$$

(10 pts) (b) Suppose that $\boldsymbol{b}$ is a vector satisfying $\begin{bmatrix} 60 & -60 & 0 & 0 \end{bmatrix}^\intercal = L^{-1}P\boldsymbol{b}$. Use this information to find all solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your reasoning to receive credit.

Solution. Saying that $\begin{bmatrix} 60 & -60 & 0 & 0 \end{bmatrix}^\intercal = L^{-1}P\boldsymbol{b}$ is equivalent to saying that $L\boldsymbol{y} = P\boldsymbol{b}$ is solved by $\boldsymbol{y} = \begin{bmatrix} 60 & -60 & 0 & 0 \end{bmatrix}^\intercal$. We may then immediately jump to solving $U\boldsymbol{x} = \boldsymbol{y}$ with back-substitution.

$$\begin{array}{rcrcrcrcrcl} 6x_1 & - & 36x_2 & + & 7x_3 & = & 60 & \to & x_1 & = & 24 + 6c_1 \\ & & & & 5x_3 & = & -60 & \to & x_3 & = & -12 \\ & & & & 0 & = & 0 & & & & \\ & & & & 0 & = & 0 & & & & \end{array} \qquad \overset{\boldsymbol{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} 6c_1 + 24 \\ c_1 \\ -12 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ -12 \end{bmatrix} + c_1 \cdot \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

(4 pts) (c) Which of the following statements accurately describes the rows and columns of A?

- ◯ The rows of A are independent and the columns of A are independent.
- ◯ The rows of A are dependent and the columns of A are independent.
- ◯ The rows of A are independent and the columns of A are dependent.
- √ The rows of A are dependent and the columns of A are dependent.

(4 pts) (d) The last two rows of one of the following matrices form a basis of $\operatorname{Null}(A^\intercal)$. Select this matrix.

√ $L^{-1}P$ ◯ L ◯ LP ◯ PL^{-1} ◯ UL

Problem 2. Suppose that A is a 3×3 matrix satisfying the following three equations.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-2\\-1\end{bmatrix} \qquad \operatorname{rref}\left[\begin{array}{ccc|c} & & & 1\\ & A & & 1\\ & & & 1\end{array}\right] = \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{array}\right] \qquad \operatorname{rref}\left[\begin{array}{ccc|c} & & & 1\\ & A^\intercal & & 1\\ & & & 1\end{array}\right] = \left[\begin{array}{ccc|c}1 & 0 & 1 & -11\\0 & 1 & 1 & -7\\0 & 0 & 0 & 0\end{array}\right]$$

Note that $\operatorname{rref}(A)$ and $\operatorname{rref}(A^\intercal)$ can be inferred from the second and third equations above.

(4 pts) (*a*) The vector $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\intercal$ belongs to exactly one of the four fundamental subspaces of A. Select this space.

○ The null space. √ The row space. ○ The column space. ○ The left null space.

(4 pts) (*b*) The vector $\begin{bmatrix}1 & -2 & -1\end{bmatrix}^\intercal$ belongs to exactly one of the four fundamental subspaces of A. Select this space.

○ The null space. ○ The row space. √ The column space. ○ The left null space.

(5 pts) (*c*) Determine if $\begin{bmatrix}1 & -1 & 1\end{bmatrix}^\intercal \in \operatorname{Null}(A)$. Clearly explain your reasoning to receive credit.

Solution. The solutions to $A\boldsymbol{x} = \boldsymbol{O}$ are the same as the solutions to $\operatorname{rref}(A)\boldsymbol{x} = \boldsymbol{O}$. The middle equation gives us $\operatorname{rref}(A)$, so we need only calculate

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}2\\-1\\0\end{bmatrix} \neq \boldsymbol{O}$$

This means that $\begin{bmatrix}1 & -1 & 1\end{bmatrix}^\intercal \notin \operatorname{Null}(A)$.

Alternatively, one may note that the nonzero rows of $\operatorname{rref}(A)$ form a basis of $\operatorname{Col}(A^\intercal) = \operatorname{Null}(A)^\perp$. The given vector is not orthogonal to either of those basis vectors so $\begin{bmatrix}1 & -1 & 1\end{bmatrix}^\intercal \notin \operatorname{Null}(A)$.

(5 pts) (*d*) Find a basis of the left null space of A. Clearly explain your reasoning to receive credit.

Solution. As discussed in class, one basis of $\operatorname{Null}(A^\intercal)$ consists of the "pivot solutions" to $A^\intercal\boldsymbol{x} = \boldsymbol{O}$. We can find these solutions by looking at $\operatorname{rref}(A^\intercal)$, which is given in the third equation.

Evidently, in $A^\intercal\boldsymbol{x} = \boldsymbol{O}$, the first two variables are dependent and the third is free. Looking at the relevant equations here gives $\operatorname{Null}(A^\intercal) = \operatorname{Span}\{\begin{bmatrix}-1 & -1 & 1\end{bmatrix}^\intercal\}$.

(7 pts) **Problem 3.** Suppose that A and B are $n \times n$ matrices and that $\boldsymbol{v} \in \mathbb{R}^n$ satisfies $\boldsymbol{v} \in \mathcal{E}_A(-2)$ and $\boldsymbol{v} \in \mathcal{E}_B(5)$. Show that $\boldsymbol{v}$ is an eigenvector of $M = A^2 + AB - I_n$ and identify the corresponding eigenvalue.

Solution. The given information tells us that $Av = -2 \cdot \boldsymbol{v}$ and $B\boldsymbol{v} = 5 \cdot \boldsymbol{v}$. It follows that

$$\begin{aligned} M\boldsymbol{v} &= (A^2 + AB - I_n)\boldsymbol{v} \\ &= A^2\boldsymbol{v} + AB\boldsymbol{v} - \boldsymbol{v} \\ &= (-2) \cdot A\boldsymbol{v} + 5 \cdot A\boldsymbol{v} - \boldsymbol{v} \\ &= (-2)^2 \cdot \boldsymbol{v} - 10 \cdot \boldsymbol{v} - \boldsymbol{v} \\ &= -7 \cdot \boldsymbol{v} \end{aligned}$$

This shows that $\boldsymbol{v} \in \mathcal{E}_M(-7)$.

Problem 4. Suppose that A is a matrix whose null space and left null space are given by

$$\operatorname{Null}(A) = \operatorname{Col} \overset{K}{\begin{bmatrix} 1 & 0 & 0 & 3 \\ 1 & 2 & 0 & 7 \\ 2 & 3 & 4 & 32 \\ 0 & 0 & 1 & 5 \\ -1 & -1 & -2 & -15 \end{bmatrix}} \qquad \operatorname{Null}(A^\intercal) = \operatorname{Col} \overset{C}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}}$$

Note here that K is 5×4 and that C is 4×2.

(6 pts) (*a*) Explain why $\{\begin{bmatrix}1 & 0 & -1 & -1\end{bmatrix}^\intercal, \begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix}^\intercal\}$ is a basis of $\operatorname{Null}(A^\intercal)$.

Solution. The second equation tells us that $\operatorname{Null}(A^\intercal) = \operatorname{Span}\{\begin{bmatrix}1 & 0 & -1 & -1\end{bmatrix}^\intercal, \begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix}^\intercal\}$. These two vectors are independent because they are not multiples of each other. Therefore, these two vectors form a basis of $\operatorname{Null}(A^\intercal)$.

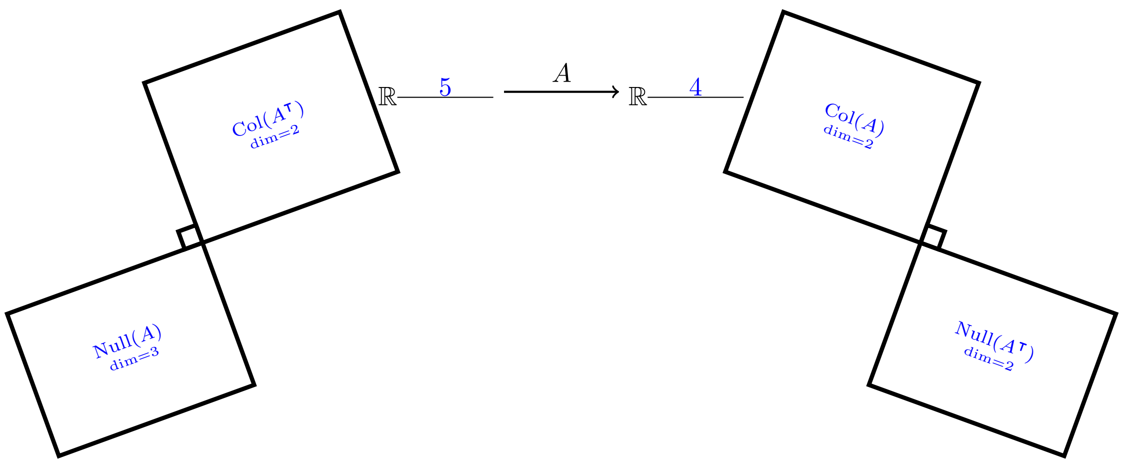
(6 pts) (*b*) Find all values of c for which $A\boldsymbol{x} = \begin{bmatrix}2 & c & -3 & 5\end{bmatrix}^\intercal$ is consistent. Clearly explain your reasoning to receive credit.

Solution. This is equivalent to requiring that $\begin{bmatrix}2 & c & -3 & 5\end{bmatrix}^\intercal \in \operatorname{Col}(A)$. Since $\operatorname{Col}(A) = \operatorname{Null}(A^\intercal)^\perp$, we must have

$$0 = \langle \begin{bmatrix}2 & c & -3 & 5\end{bmatrix}^\intercal, \begin{bmatrix}1 & 0 & -1 & -1\end{bmatrix}^\intercal \rangle = 0 \qquad 0 = \langle \begin{bmatrix}2 & c & -3 & 5\end{bmatrix}^\intercal, \begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix}^\intercal \rangle = c + 2$$

The only meaningful equation here is $0 = c + 2$, which gives $c = -2$.

(10 pts) (*c*) Fill in every missing label in the picture of the four fundamental subspaces below, including the dimension of each fundamental subspace.



Solution. We know that $\operatorname{Null}(A) = \operatorname{Col}(K) \subset \mathbb{R}^5$ because K is 5×4. We also know that $\operatorname{Null}(A^\intercal) = \operatorname{Col}(C) \subset \mathbb{R}^4$ because C is 4×2. We are told in part (*a*) that $\{\begin{bmatrix}1 & 0 & -1 & -1\end{bmatrix}^\intercal, \begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix}^\intercal\}$ is a basis of $\operatorname{Null}(A^\intercal)$ which means that $\dim \operatorname{Null}(A^\intercal) = 2$. The rest of the dimensions follow from the rank-nullity theorem.

(5 pts) (*d*) Does K have independent or dependent columns? Clearly explain your reasoning to receive credit.

Solution. We know that $\operatorname{rank}(K) = \dim \operatorname{Col}(K) = \dim \operatorname{Null}(A) = 3$ from our picture. But K has 4 which means K does not have full column rank. This means that the columns of K are dependent.

Problem 5. Suppose that A is a 4×3 matrix and that the projection matrix P onto $\operatorname{Col}(A)$ satisfies the following equations.

$$\begin{bmatrix} & P & \end{bmatrix}\begin{bmatrix}-1\\-3\\-1\\0\end{bmatrix}=\begin{bmatrix}-1\\-3\\-1\\0\end{bmatrix} \quad \begin{bmatrix} & P & \end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix} & P & \end{bmatrix}\begin{bmatrix}-1\\-8\\0\\0\end{bmatrix}=\begin{bmatrix}-1\\-8\\0\\0\end{bmatrix} \quad \begin{bmatrix} & P & \end{bmatrix}\begin{bmatrix}2\\1\\1\\-4\end{bmatrix}=\begin{bmatrix}2\\1\\1\\0\end{bmatrix}$$

(6 pts) (a) Which of the following vectors belongs to $\mathcal{E}_P(1)$? Select all that apply.

$\surd$ $\begin{bmatrix}-1 & -3 & -1 & 0\end{bmatrix}^\intercal$ ◯ $\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^\intercal$ $\surd$ $\begin{bmatrix}-1 & -8 & 0 & 0\end{bmatrix}^\intercal$ ◯ $\begin{bmatrix}2 & 1 & 1 & -4\end{bmatrix}^\intercal$

(6 pts) (b) Which of the following vectors belongs to $\operatorname{Null}(A^\intercal)$? Select all that apply.

◯ $\begin{bmatrix}-1 & -3 & -1 & 0\end{bmatrix}^\intercal$ $\surd$ $\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^\intercal$ ◯ $\begin{bmatrix}-1 & -8 & 0 & 0\end{bmatrix}^\intercal$ ◯ $\begin{bmatrix}2 & 1 & 1 & -4\end{bmatrix}^\intercal$

(6 pts) (c) Find the projection of $\boldsymbol{b} = \begin{bmatrix}2 & 1 & 1 & -4\end{bmatrix}^\intercal$ onto $\operatorname{Null}(A^\intercal)$. Clearly explain your reasoning to receive credit.

Solution. The orthogonality conditions say that

$$P_{\operatorname{Null}(A^\intercal)}\boldsymbol{b} = (I-P)\boldsymbol{b} = \boldsymbol{b} - P\boldsymbol{b} = \begin{bmatrix}2\\1\\1\\-4\end{bmatrix} - \begin{bmatrix}2\\1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\-4\end{bmatrix}$$

(5 pts) (d) Find the error E in using the technique of least squares to approximate a solution to the system $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = \begin{bmatrix}2 & 1 & 1 & -4\end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit.

Solution. By definition, we have

$$E = \|\boldsymbol{b} - A\widehat{x}\|^2 = \|\boldsymbol{b} - P\boldsymbol{b}\|^2 = \left\| \begin{bmatrix}2\\1\\1\\-4\end{bmatrix} - \begin{bmatrix}2\\1\\1\\0\end{bmatrix} \right\|^2 = 16$$