DUKE UNIVERSITY

MATH 218D-2

MATRICES AND VECTORS

Exam II

Name: ____________________ *NetID:* ____________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ____________________

March 4, 2022

- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. The following row-reductions calculate the matrix U in $PA = LU$.

$$\overset{A}{\begin{bmatrix} 6 & -36 & 7 \\ 12 & -72 & 14 \\ -18 & 108 & -16 \\ -6 & 36 & 13 \end{bmatrix}} \xrightarrow{\substack{r_2 - 2\cdot r_1 \to r_2 \\ r_3 + 3\cdot r_1 \to r_3 \\ r_4 + r_1 \to r_4}} \begin{bmatrix} 6 & -36 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 20 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 6 & -36 & 7 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 20 \end{bmatrix} \xrightarrow{r_4 - 4\cdot r_2 \to r_4} \overset{U}{\begin{bmatrix} 6 & -36 & 7 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

(7 pts) (a) Find L.

(10 pts) (b) Suppose that $\boldsymbol{b}$ is a vector satisfying $\begin{bmatrix} 60 & -60 & 0 & 0 \end{bmatrix}^\intercal = L^{-1}P\boldsymbol{b}$. Use this information to find all solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your reasoning to receive credit.

(4 pts) (c) Which of the following statements accurately describes the rows and columns of A?

- ○ The rows of A are independent and the columns of A are independent.
- ○ The rows of A are dependent and the columns of A are independent.
- ○ The rows of A are independent and the columns of A are dependent.
- ○ The rows of A are dependent and the columns of A are dependent.

(4 pts) (d) The last two rows of one of the following matrices form a basis of $\operatorname{Null}(A^\intercal)$. Select this matrix.

○ $L^{-1}P$ ○ L ○ LP ○ PL^{-1} ○ UL

Problem 2. Suppose that A is a 3×3 matrix satisfying the following three equations.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\-2\\-1\end{bmatrix} \qquad \operatorname{rref}\left[\begin{array}{ccc|c} & & & 1\\ & A & & 1\\ & & & 1\end{array}\right]=\left[\begin{array}{ccc|c}1&0&1&0\\0&1&0&0\\0&0&0&1\end{array}\right] \qquad \operatorname{rref}\left[\begin{array}{ccc|c} & & & 1\\ & A^{\intercal} & & 1\\ & & & 1\end{array}\right]=\left[\begin{array}{ccc|c}1&0&1&-11\\0&1&1&-7\\0&0&0&0\end{array}\right]$$

Note that $\operatorname{rref}(A)$ and $\operatorname{rref}(A^{\intercal})$ can be inferred from the second and third equations above.

(4 pts) (*a*) The vector $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^{\intercal}$ belongs to exactly one of the four fundamental subspaces of A. Select this space.

○ The null space. ○ The row space. ○ The column space. ○ The left null space.

(4 pts) (*b*) The vector $\begin{bmatrix}1 & -2 & -1\end{bmatrix}^{\intercal}$ belongs to exactly one of the four fundamental subspaces of A. Select this space.

○ The null space. ○ The row space. ○ The column space. ○ The left null space.

(5 pts) (*c*) Determine if $\begin{bmatrix}1 & -1 & 1\end{bmatrix}^{\intercal} \in \operatorname{Null}(A)$. Clearly explain your reasoning to receive credit.

(5 pts) (*d*) Find a basis of the left null space of A. Clearly explain your reasoning to receive credit.

(7 pts) **Problem 3.** Suppose that A and B are $n\times n$ matrices and that $\boldsymbol{v}\in\mathbb{R}^n$ satisfies $\boldsymbol{v}\in\mathcal{E}_A(-2)$ and $\boldsymbol{v}\in\mathcal{E}_B(5)$. Show that $\boldsymbol{v}$ is an eigenvector of $M=A^2+AB-I_n$ and identify the corresponding eigenvalue.

Problem 4. Suppose that A is a matrix whose null space and left null space are given by

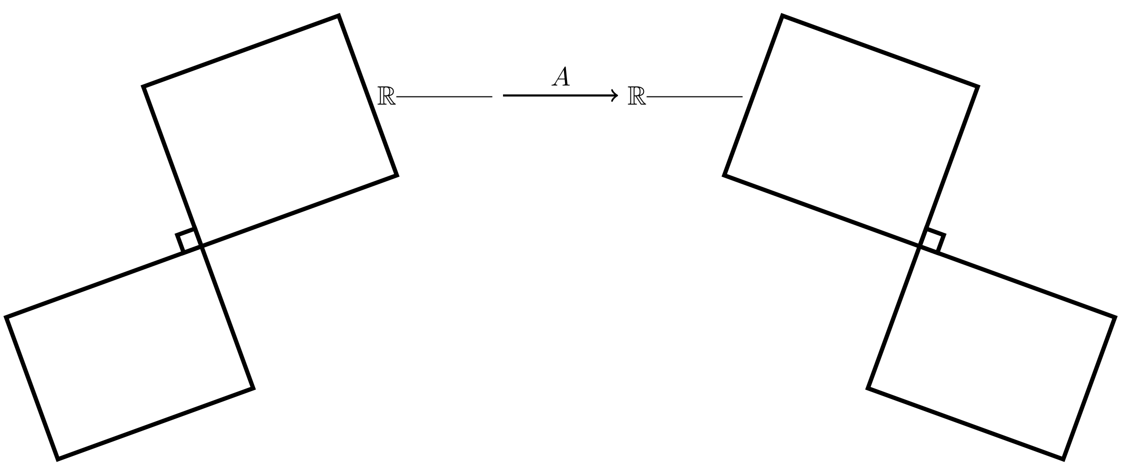
$$\operatorname{Null}(A)=\operatorname{Col}\overset{K}{\begin{bmatrix}1&0&0&3\\1&2&0&7\\2&3&4&32\\0&0&1&5\\-1&-1&-2&-15\end{bmatrix}}\qquad\qquad \operatorname{Null}(A^\intercal)=\operatorname{Col}\overset{C}{\begin{bmatrix}1&0\\0&1\\-1&1\\-1&1\end{bmatrix}}$$

Note here that K is 5×4 and that C is 4×2.

(6 pts) (*a*) Explain why $\{\begin{bmatrix}1&0&-1&-1\end{bmatrix}^\intercal, \begin{bmatrix}0&1&1&1\end{bmatrix}^\intercal\}$ is a basis of $\operatorname{Null}(A^\intercal)$.

(6 pts) (*b*) Find all values of c for which $A\boldsymbol{x}=\begin{bmatrix}2&c&-3&5\end{bmatrix}^\intercal$ is consistent. Clearly explain your reasoning to receive credit.

(10 pts) (*c*) Fill in every missing label in the picture of the four fundamental subspaces below, including the dimension of each fundamental subspace.



(5 pts) (*d*) Does K have independent or dependent columns? Clearly explain your reasoning to receive credit.

Problem 5. Suppose that A is a 4×3 matrix and that the projection matrix P onto $\operatorname{Col}(A)$ satisfies the following equations.

$$\begin{bmatrix} & P & \end{bmatrix}\begin{bmatrix}-1\\-3\\-1\\0\end{bmatrix}=\begin{bmatrix}-1\\-3\\-1\\0\end{bmatrix} \quad \begin{bmatrix} & P & \end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix} & P & \end{bmatrix}\begin{bmatrix}-1\\-8\\0\\0\end{bmatrix}=\begin{bmatrix}-1\\-8\\0\\0\end{bmatrix} \quad \begin{bmatrix} & P & \end{bmatrix}\begin{bmatrix}2\\1\\1\\-4\end{bmatrix}=\begin{bmatrix}2\\1\\1\\0\end{bmatrix}$$

(6 pts) (a) Which of the following vectors belongs to $\mathcal{E}_P(1)$? Select all that apply.

○ $\begin{bmatrix}-1 & -3 & -1 & 0\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}-1 & -8 & 0 & 0\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}2 & 1 & 1 & -4\end{bmatrix}^\intercal$

(6 pts) (b) Which of the following vectors belongs to $\operatorname{Null}(A^\intercal)$? Select all that apply.

○ $\begin{bmatrix}-1 & -3 & -1 & 0\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}-1 & -8 & 0 & 0\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}2 & 1 & 1 & -4\end{bmatrix}^\intercal$

(6 pts) (c) Find the projection of $\boldsymbol{b} = \begin{bmatrix}2 & 1 & 1 & -4\end{bmatrix}^\intercal$ onto $\operatorname{Null}(A^\intercal)$. Clearly explain your reasoning to receive credit.

(5 pts) (d) Find the error E in using the technique of least squares to approximate a solution to the system $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = \begin{bmatrix}2 & 1 & 1 & -4\end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit.