Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: Solutions

NetID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

April 8, 2022

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

Problem 1. Suppose that $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3 \in \mathbb{R}^4$ are mutually orthogonal vectors satisfying $\|\boldsymbol{w}_1\| = \|\boldsymbol{w}_2\| = \|\boldsymbol{w}_3\| = c$ where $c > 0$. Further suppose that A factors as $A = WR$ where

$$W = \begin{bmatrix} | & | & | \\ \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 \\ | & | & | \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(5 pts) (*a*) Find $W^\intercal W$ (note that this matrix depends on the scalar c).

Solution. The Gramian construction allows us to explicitly calculate $W^\intercal W$ as a matrix of inner products.

$$W^\intercal W = \begin{bmatrix} \langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle & \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle & \langle \boldsymbol{w}_1, \boldsymbol{w}_3 \rangle \\ \langle \boldsymbol{w}_2, \boldsymbol{w}_1 \rangle & \langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle & \langle \boldsymbol{w}_2, \boldsymbol{w}_3 \rangle \\ \langle \boldsymbol{w}_3, \boldsymbol{w}_1 \rangle & \langle \boldsymbol{w}_3, \boldsymbol{w}_2 \rangle & \langle \boldsymbol{w}_3, \boldsymbol{w}_3 \rangle \end{bmatrix} = \begin{bmatrix} \|\boldsymbol{w}_1\|^2 & 0 & 0 \\ 0 & \|\boldsymbol{w}_2\|^2 & 0 \\ 0 & 0 & \|\boldsymbol{w}_3\|^2 \end{bmatrix} = \begin{bmatrix} c^2 & & \\ & c^2 & \\ & & c^2 \end{bmatrix} = c^2 \cdot I_3$$

(5 pts) (*b*) Show that $A^\intercal A = c^2 \cdot R^\intercal R$.

Solution. $A^\intercal A = (WR)^\intercal (WR) = R^\intercal W^\intercal W R = R^\intercal (c^2 \cdot I_3) R = c^2 \cdot R^\intercal R$

(8 pts) (*c*) Suppose that $\boldsymbol{b}$ is a vector satisfying $W^\intercal \boldsymbol{b} = c^4 \cdot \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^\intercal$. Find the least squares approximate solution $\widehat{\boldsymbol{x}}$ to $A\boldsymbol{x} = \boldsymbol{b}$ (note that $\widehat{x}$ depends on the scalar c).

Solution. The least-squares problem is $A^\intercal A \widehat{x} = A^\intercal \boldsymbol{b}$, which reduces to $c^2 \cdot R^\intercal R \widehat{x} = R^\intercal W^\intercal \boldsymbol{b} = c^4 \cdot R^\intercal \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^\intercal$. Since R is nonsingular, we can further reduce to $R\widehat{x} = c^2 \cdot \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^\intercal$ and solve with back-substitution.

$$\begin{array}{rcrcrcrcrcrcr} \widehat{x}_1 & + & \widehat{x}_2 & + & \widehat{x}_3 & = & c^2 & \to & \widehat{x}_1 & = & c^2 - (-2\,c^2) - c^2 & = & 2\,c^2 \\ & & \widehat{x}_2 & + & \widehat{x}_3 & = & -c^2 & \to & \widehat{x}_2 & = & -c^2 - c^2 & = & -2\,c^2 \\ & & & & \widehat{x}_3 & = & c^2 & & & & & & \end{array}$$

This gives $\widehat{x} = \begin{bmatrix} 2\,c^2 & -2\,c^2 & c^2 \end{bmatrix}^\intercal$.

(7 pts) (*d*) Is R diagonalizable? Explain why or why not.

Solution. Note that R is upper-triangular and the only value on the diagonal is 1. This means that $\lambda = 1$ is the only eigenvalue of R and its geometric multiplicity is

$$\operatorname{gm}_R(1) = \operatorname{nullity} \overbrace{\begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}^{1 \cdot I_3 - R} = 1 \neq 3$$

This means that R is *not diagonalizable*.

Problem 2. Consider the nonsingular matrix $A = \begin{bmatrix} 1 & i & i & 0 \\ -1 & -3i+1 & 1 & 1 \\ -1 & -i & -i & i \\ -1 & -i & 4i & 7 \end{bmatrix}$.

(13 pts) (*a*) Find $\det(A)$.

Solution. This is a large matrix, so our best chance at calculating $\det(A)$ is to row-reduce.

$$\begin{vmatrix} 1 & i & i & 0 \\ -1 & -3i+1 & 1 & 1 \\ -1 & -i & -i & i \\ -1 & -i & 4i & 7 \end{vmatrix} \xlongequal[r_4+r_1\to r_4]{\substack{r_2+r_1\to r_2 \\ r_3+r_1\to r_3}} \begin{vmatrix} 1 & i & i & 0 \\ 0 & -2i+1 & i+1 & 1 \\ 0 & 0 & 0 & i \\ 0 & 0 & 5i & 7 \end{vmatrix} \xlongequal{r_3\leftrightarrow r_4} -\begin{vmatrix} 1 & i & i & 0 \\ 0 & -2i+1 & i+1 & 1 \\ 0 & 0 & 5i & 7 \\ 0 & 0 & 0 & i \end{vmatrix} = -10i+5$$

(12 pts) (*b*) Find the $(1,4)$ entry of $\det(A)\cdot A^{-1}$.

Solution. The adjugate formula for inverses says $A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$, so $\det(A)\cdot A^{-1} = \operatorname{adj}(A)$. This means that the $(1,4)$ entry of $\det(A)\cdot A^{-1}$ is the $(1,4)$ entry of $\operatorname{adj}(A)$, which the $(4,1)$ cofactor of A. Our desired value is then

$$(-1)^{4+1}\cdot \begin{vmatrix} i & i & 0 \\ -3i+1 & 1 & 1 \\ -i & -i & i \end{vmatrix} = -i^2 \cdot \begin{vmatrix} 1 & 1 & 0 \\ -3i+1 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ -3i+1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 3i$$

Problem 3. Suppose that A is a matrix whose characteristic polynomial is given by

$$\chi_A(t) = t^6 - 6\,t^4 - 4\,t^3 + 9\,t^2 + 12\,t + 4$$

(8 pts) (a) trace(A) = $\underline{\quad 0 \quad}$ and det(A) = $\underline{\quad 4 \quad}$

(9 pts) (b) Does $(I - A)^{-1}$ exist? Clearly explain why or why not.

Solution. This is equivalent to asking if $I - A$ is nonsingular, which is equivalent to asking if $\lambda = 1$ is *not* an eigenvalue of A, which is equivalent to asking if $\lambda = 1$ is *not* a root of $\chi_A(t)$, which is resolved with the calculation

$$\chi_A(1) = 1 - 6 - 4 + 9 + 12 + 4 = 16 \neq 0$$

Evidently, $\lambda = 1$ is *not* and eigenvalue of A, so $(I - A)^{-1}$ does exist..

(8 pts) (c) If possible, find rank(A). If this is not possible then explain why.

Solution. Note that $\chi_A(0) = 4 \neq 0$, so $\lambda = 0$ is not an eigenvalue of A. This means that A is nonsingular, so $\operatorname{rank}(A) = \deg(\chi_A) = 6$.

Alternatively, $\det(A) = (-1)^{\deg(\chi_A)} \cdot \chi_A(0) = 4 \neq 0$. This means that A is nonsingular so $\operatorname{rank}(A) = \deg(\chi_A) = 6$.

Problem 4. Suppose that $A = XDX^{-1}$ where

$$A = \begin{bmatrix} * & -235 & 71 & -237 \\ * & 19 & -4 & 12 \\ * & 435 & -112 & 357 \\ * & 40 & -8 & 23 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 1 & 8 & -24 \\ 0 & 1 & 1 & -4 \\ -1 & 9 & 15 & -65 \\ 0 & 2 & 6 & -23 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Note that the first column of A is currently unknown.

(5 pts) (*a*) Find the complete factorization of $\chi_A(t)$. Clearly explain your reasoning to receive credit.

Solution. We are gifted a diagonalization of A, so the roots of $\chi_A(t)$ are sitting on the diagonal of A. Hence

$$\chi_A(t) = (t-2)\cdot(t-3)\cdot(t+1)^2$$

(10 pts) (*b*) Find the missing column of A. Clearly explain your reasoning to receive credit.
Hint. Note that $\boldsymbol{v} = \begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}^\intercal$ is the first column of X.

Solution. We know that $\boldsymbol{v} \in \mathcal{E}_A(2)$, which means that $A\boldsymbol{v} = 2\cdot\boldsymbol{v}$. Taking advantage of the nice coordinates $\boldsymbol{v}$ then gives us the equation

$$\mathrm{Col}_1(A) - \mathrm{Col}_3(A) = 2\cdot\boldsymbol{v}$$

Solving for $\mathrm{Col}_1(A)$ gives

$$\mathrm{Col}_1(A) = 2\cdot\boldsymbol{v} + \mathrm{Col}_3(A) = \begin{bmatrix}2 & 0 & -2 & 0\end{bmatrix}^\intercal + \begin{bmatrix}71 & -4 & -112 & -8\end{bmatrix}^\intercal = \begin{bmatrix}73 & -4 & -114 & -8\end{bmatrix}^\intercal$$

(10 pts) (*c*) Suppose that $\boldsymbol{u}_0 \in \mathbb{R}^4$ satisfies $X^{-1}\boldsymbol{u}_0 = \begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}^\intercal$ and that $\boldsymbol{u}(t)$ solves the initial value problem $\boldsymbol{u}' = A\boldsymbol{u}$ with $\boldsymbol{u}(0) = \boldsymbol{u}_0$. Which, if any, of the coordinates of $\boldsymbol{u}(t)$ tend to zero as $t \to \infty$? Clearly explain your reasoning to receive credit.

Solution. Here we have

$$\boldsymbol{u}(t) = \exp(At)\boldsymbol{u}_0 = X\overbrace{\begin{bmatrix} e^{2\,t} & & & \\ & e^{3\,t} & & \\ & & e^{-t} & \\ & & & e^{-t} \end{bmatrix}}^{\exp(Dt)}\overbrace{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}}^{X^{-1}\boldsymbol{u}_0} = \begin{bmatrix} 1 & 1 & 8 & -24 \\ 0 & 1 & 1 & -4 \\ -1 & 9 & 15 & -65 \\ 0 & 2 & 6 & -23 \end{bmatrix}\begin{bmatrix} e^{2\,t} \\ 0 \\ e^{-t} \\ 0 \end{bmatrix} = \begin{bmatrix} e^{2\,t} + 8\,e^{-t} \\ e^{-t} \\ -e^{2\,t} + 15\,e^{-t} \\ 6\,e^{-t} \end{bmatrix}$$

Here, we find that the second and fourth coordinates of $\boldsymbol{u}(t)$ tend to zero as $t \to \infty$ while the first and third coordinates tend to infinity.