DUKE UNIVERSITY

MATH 218D-2

MATRICES AND VECTORS

Exam III

Name: ______________________ *NetID:* ______________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

April 8, 2022

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Suppose that $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3 \in \mathbb{R}^4$ are mutually orthogonal vectors satisfying $\|\boldsymbol{w}_1\| = \|\boldsymbol{w}_2\| = \|\boldsymbol{w}_3\| = c$ where $c > 0$. Further suppose that A factors as $A = WR$ where

$$W = \begin{bmatrix} | & | & | \\ \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 \\ | & | & | \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(5 pts) (a) Find $W^\intercal W$ (note that this matrix depends on the scalar c).

(5 pts) (b) Show that $A^\intercal A = c^2 \cdot R^\intercal R$.

(8 pts) (c) Suppose that $\boldsymbol{b}$ is a vector satisfying $W^\intercal \boldsymbol{b} = c^4 \cdot \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^\intercal$. Find the least squares approximate solution $\widehat{\boldsymbol{x}}$ to $A\boldsymbol{x} = \boldsymbol{b}$ (note that $\widehat{x}$ depends on the scalar c).

(7 pts) (d) Is R diagonalizable? Explain why or why not.

Problem 2. Consider the nonsingular matrix $A = \begin{bmatrix} 1 & i & i & 0 \\ -1 & -3i+1 & 1 & 1 \\ -1 & -i & -i & i \\ -1 & -i & 4i & 7 \end{bmatrix}$.

(13 pts) (a) Find $\det(A)$.

(12 pts) (b) Find the $(1,4)$ entry of $\det(A) \cdot A^{-1}$.

Problem 3. Suppose that A is a matrix whose characteristic polynomial is given by

$$\chi_A(t) = t^6 - 6\,t^4 - 4\,t^3 + 9\,t^2 + 12\,t + 4$$

(8 pts) (a) trace$(A) =$ ________ and det$(A) =$ ________

(9 pts) (b) Does $(I - A)^{-1}$ exist? Clearly explain why or why not.

(8 pts) (c) If possible, find rank(A). If this is not possible then explain why.

Problem 4. Suppose that $A = XDX^{-1}$ where

$$A = \begin{bmatrix} * & -235 & 71 & -237 \\ * & 19 & -4 & 12 \\ * & 435 & -112 & 357 \\ * & 40 & -8 & 23 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 1 & 8 & -24 \\ 0 & 1 & 1 & -4 \\ -1 & 9 & 15 & -65 \\ 0 & 2 & 6 & -23 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Note that the first column of A is currently unknown.

(5 pts) (a) Find the complete factorization of $\chi_A(t)$. Clearly explain your reasoning to receive credit.

(10 pts) (b) Find the missing column of A. Clearly explain your reasoning to receive credit.
Hint. Note that $\boldsymbol{v} = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^\intercal$ is the first column of X.

(10 pts) (c) Suppose that $\boldsymbol{u}_0 \in \mathbb{R}^4$ satisfies $X^{-1}\boldsymbol{u}_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^\intercal$ and that $\boldsymbol{u}(t)$ solves the initial value problem $\boldsymbol{u}' = A\boldsymbol{u}$ with $\boldsymbol{u}(0) = \boldsymbol{u}_0$. Which, if any, of the coordinates of $\boldsymbol{u}(t)$ tend to zero as $t \to \infty$? Clearly explain your reasoning to receive credit.