Math 218D: Week 1 Discussion

Study Copy

August 31, 2023

Problem 1. Fill in the blanks below.

$$A=\begin{bmatrix}3 & -4 & 2 & 8 & 0\\ 1 & 8 & 1/9 & 0 & 1\\ 2 & 4 & -1 & \pi & 2\end{bmatrix}\qquad a_{23}=\_\_\_\_ \qquad \operatorname{Col}_2(A)\in\mathbb{R}^{\_\_\_\_}\qquad A^\intercal=\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Problem 2. Fill in the blanks below, assuming that S is *symmetric*.

$$S=\begin{bmatrix}5 & -4 & \_\_\_ & \_\_\_\\ \_\_\_ & 19 & \_\_\_ & -1\\ 11 & 2 & 8 & 3\\ 9 & \_\_\_ & \_\_\_ & -10\end{bmatrix}\qquad \operatorname{trace}(S)=\_\_\_\_$$

Problem 3. By definition, a matrix S is *symmetric* if ______________.

Problem 4. Suppose that A is $n\times n$ and let $S=A+A^\intercal$. Prove that S is symmetric.

Hint. This proof can be quickly accomplished by filling in the blanks below.

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Problem 5. It is known that the columns of the matrix A below satisfy the equation on the right.

$$A=\begin{bmatrix}6 & 11 & 17 & *\\ 14 & 8 & 3 & *\\ 9 & 13 & 4 & *\end{bmatrix}\qquad 2\cdot\operatorname{Col}_1-\operatorname{Col}_2+3\cdot\operatorname{Col}_3-\operatorname{Col}_4=\boldsymbol{O}$$

Use this information to find the missing column of A.

Problem 6. Find the missing entries in $A=\begin{bmatrix}\_\_\_ & 2 & -1\\ -12 & -4 & 2\\ \_\_\_ & 10 & -5\\ \_\_\_ & 8 & -4\end{bmatrix}$ assuming A has rank one.

Problem 7. Consider the matrix R given by

$$R = \begin{bmatrix} 1 & -3 & 0 & -9 & 5 \\ 0 & 0 & 1 & 14 & 9 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad\qquad \operatorname{Col}_4(R) = \underline{\qquad} \operatorname{Col}_1(R) + \underline{\qquad} \operatorname{Col}_3(R)$$

(a) Fill in the blanks above to express the fourth column of R as a linear combination of the first and third columns of R.

(b) Can the fifth column of R be expressed as a linear combination of the first and third columns of R? Explain why or why not.

Problem 8. We write $\mathbb{R}^9 \xrightarrow{A} \mathbb{R}^{22}$ to indicate that A is a $\underline{\qquad} \times \underline{\qquad}$ matrix.

Problem 9. Fill in the blanks in the two equations below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \underline{\qquad} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} + \underline{\qquad} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} + \underline{\qquad} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} \qquad \begin{bmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = 11 \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} - 42 \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}$$

Problem 10. Fill in the blanks in each equation below.

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \text{the third column of } A \qquad \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \text{the first column minus the third column of } A$$

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \text{the sum of all columns of } A \qquad \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \text{twice the first column of } A$$

Problem 11. Suppose that A has four columns. Fill in the blanks in the equation on the right to validate the equation on the left.

$$\operatorname{Col}_1 + 3 \operatorname{Col}_2 - 9 \operatorname{Col}_3 = 6 \operatorname{Col}_4 \qquad\qquad \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \boldsymbol{O}$$

Problem 12. We say that $\boldsymbol{v}$ is an *eigenvector* of A with *corresponding eigenvalue* λ if $A\boldsymbol{v} = \underline{\qquad\qquad\qquad}$.

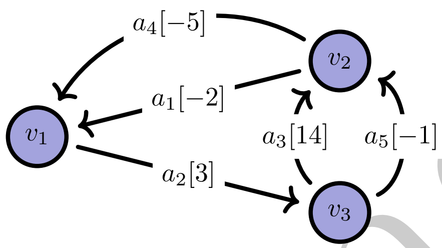
Problem 13. Suppose that A is $n \times n$ and that $\boldsymbol{v} \in \mathcal{E}_A(\lambda)$. Calculate $(\lambda \cdot I_n - A)\boldsymbol{v}$.

Math 218D: Week 2 Discussion

Study Copy

September 7, 2023

Problem 1. Consider the weighted digraph G depicted below.



(a) The number of connected components of G is ________.

(b) The circuit rank of G is ________.

(c) The Euler characteristic of G is $\chi(G) =$ ________.

(d) Use a matrix-vector product to calculate the weighted net flow through each node of G.

Problem 2. $\langle \begin{bmatrix} 1 & -3 & 0 & 2 \end{bmatrix}^\intercal, \begin{bmatrix} 2 & 1 & 5 & 0 \end{bmatrix}^\intercal \rangle =$ ____

Problem 3. Which of the following vectors is *orthogonal* to $\boldsymbol{v} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\intercal$?

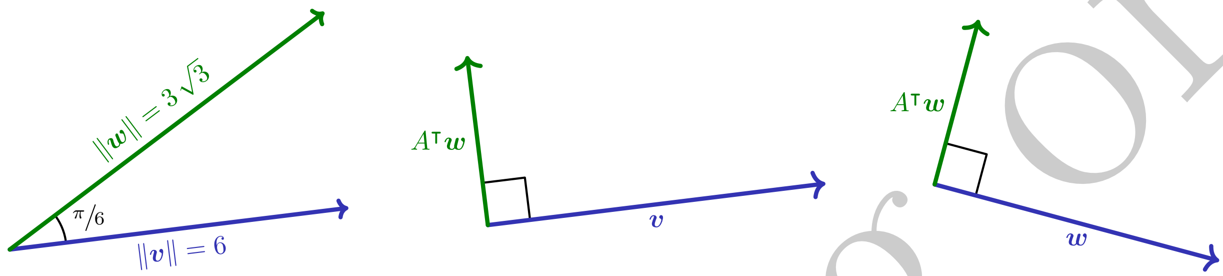
○ $\boldsymbol{w} = \begin{bmatrix} 3 & -5 & 2 & 1 & 0 \end{bmatrix}^\intercal$ ○ $\boldsymbol{x} = \begin{bmatrix} 9 & 2 & 3 & -6 & -8 \end{bmatrix}^\intercal$ ○ $\boldsymbol{y} = \begin{bmatrix} -1 & -1 & 9 & -10 & 3 \end{bmatrix}^\intercal$

Problem 4. The length of $\boldsymbol{v}$ can be calculated with an inner product using the formula $\|\boldsymbol{v}\| =$ ________________.

Problem 5. The inner product can be interpreted geometrically with the formula $\langle \boldsymbol{v}, \boldsymbol{w} \rangle =$ ________________________.

Problem 6. The adjoint formula for inner products states that $\langle A\boldsymbol{v}, \boldsymbol{w} \rangle =$ ________________________.

Problem 7. Suppose that A and B are matrices satisfying $A^{\intercal} B = I_n$ and that $\boldsymbol{v}$ and $\boldsymbol{w}$ vectors making the following diagrams accurate.



Calculate $\langle B\boldsymbol{v} - 3\,\boldsymbol{w}, 2\,A\boldsymbol{v} - A\boldsymbol{w} \rangle$.

Math 218D: Week 3 Discussion

Study Copy

September 14, 2023

Problem 1. If we view $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ as $n \times 1$ matrices, then $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ can be calculated using matrix multiplication with the formula $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \underline{\hspace{4cm}}$. This is an extremely important formula!

Problem 2. One of the following calculations is possible and the other is not. Carry out the possible calculation.

$$\begin{bmatrix} 5 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 1 \\ 3 & -3 \end{bmatrix} = \underline{\hspace{5cm}} \qquad \begin{bmatrix} 5 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 3 \\ 1 & 1 & -3 \end{bmatrix} = \underline{\hspace{5cm}}$$

Problem 3. Fill in the blanks in each of the following two equations.

$$\begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \end{bmatrix} \begin{bmatrix} 0 & -3 & 1 & 0 \\ 1 & 9 & 0 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 50 & 5 & -4 \\ 12 & 193 & -3 & 19 \end{bmatrix} \qquad \begin{bmatrix} \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 2 & -5 \\ 0 & 9 \end{bmatrix}$$

Problem 4. Suppose that A and B are 2023×2023. Prove that $S = B^{\intercal} A + A^{\intercal} B$ is symmetric.

Problem 5. The last column of a matrix A is $\begin{bmatrix} 0 & 3 & 4 \end{bmatrix}^{\intercal}$ and the Gramian of A is

$$G = \begin{bmatrix} 9 & \underline{\quad} & -6 & \underline{\quad} \\ 3 & 14 & 13 & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & 29 & \underline{\quad} \\ 5 & 13 & 0 & \underline{\quad} \end{bmatrix} = \underline{\hspace{2.5cm}}$$

(a) Fill in the missing entries of G and fill in the formula used to calculate G.

(b) The number of rows of A is $\underline{\quad}$ and the number of columns of A is $\underline{\quad}$.

(c) Which (if any) of the columns of A is orthogonal to the third column of A?

Problem 6. Suppose $R = \begin{bmatrix} 3 & 2 & 5 & 1 \\ 0 & 7 & 9 & 2 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & * & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon form and that $\boldsymbol{b}$ is a vector making $R\boldsymbol{x} = \boldsymbol{b}$ *inconsistent.*

(a) The $(4, 3)$ entry of R (marked $*$) must equal ________.

(b) $\text{rank}(R) =$ ________, $\text{nullity}(R) =$ ________, and $\text{nullity}(R^\intercal) =$ ________

(c) Consider the augmented matrix $M = [R \mid \boldsymbol{b}]$ (so M is 5×5). Then $\text{rank}(M) =$ ________.

Problem 7. Suppose we represent the system of equations below with an augmented matrix $M = [A \mid \boldsymbol{b}]$.

$$\begin{array}{ccccccccc} x_1 & + & 2\,x_2 & + & 5\,x_3 & + & 4\,x_4 & = & 1 \\ & & 3\,x_2 & & & + & 5\,x_4 & = & 6 \\ & & & & & & 7\,x_4 & = & 21 \end{array}$$

(a) The $(1, 3)$ entry of A is ________ and the $(3, 1)$ entry of A is ________.

(b) Select all of the variables in the system $A\boldsymbol{x} = \boldsymbol{b}$ that are *free.* ◯ x_1 ◯ x_2 ◯ x_3 ◯ x_4

(c) Only some of the following statements are true. Select these statements.
◯ M is in row echelon form ◯ M is in reduced row echelon form ◯ $\text{rank}(A) = \text{rank}(M)$
◯ $A\boldsymbol{x} = \boldsymbol{b}$ is consistent

(d) Use techniques from class to find the general solution $\boldsymbol{x} = \boldsymbol{x}_p + c_1 \cdot \boldsymbol{x}_1 + \cdots + c_k \cdot \boldsymbol{x}_k$ to $A\boldsymbol{x} = \boldsymbol{b}$.

Math 218D: Week 4 Discussion

Study Copy

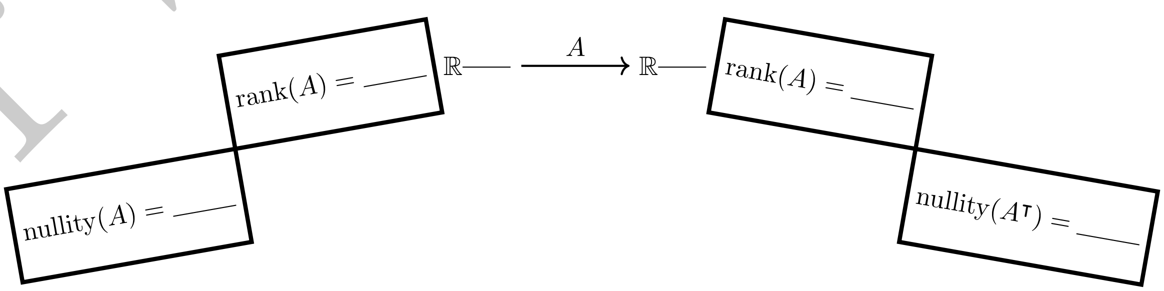
September 21, 2023

Problem 1. Consider the system of equations given by

$$\begin{array}{rcrcrcrcr} x_1 & + & 2\,x_2 & - & 4\,x_3 & + & 9\,x_4 & = & -2 \\ 5\,x_1 & + & 11\,x_2 & - & 13\,x_3 & + & 37\,x_4 & = & 7 \\ -\,3\,x_1 & - & 6\,x_2 & + & 12\,x_3 & - & 24\,x_4 & = & 0 \end{array}$$

Use the Gauß-Jordan algorithm to find the general solution to this system.

Problem 2. Suppose A is a matrix satisfying $\operatorname{rref}(A) = \begin{bmatrix} 1 & -13 & 0 & 6 \\ 0 & 0 & 1 & 17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Fill-in the blanks below.



Problem 3. Use the Gauß-Jordan algorithm to calculate $\operatorname{rref}(A)$ where $A = \begin{bmatrix} 3 & -6 & 12 & 0 & -9 \\ -7 & 14 & -28 & -5 & 26 \\ 5 & -12 & 12 & 2 & -13 \\ 2 & -3 & 12 & -3 & -5 \end{bmatrix}$.

Problem 4. Consider the $EA = R$ factorization and the vector $\boldsymbol{b}$ given by

$$\overset{E}{\begin{bmatrix} -6 & 5 & 2 & -13 \\ 4 & -4 & -2 & 9 \\ -9 & 9 & 4 & -21 \\ 1 & -2 & -1 & 3 \end{bmatrix}} \overset{A}{\begin{bmatrix} -3 & 21 & 2 & -26 & -5 & 3 \\ 3 & -21 & 1 & 14 & 38 & -2 \\ -3 & 21 & -3 & -6 & -60 & 1 \\ 2 & -14 & -1 & 16 & 7 & -2 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & -7 & 0 & 6 & 9 & 0 \\ 0 & 0 & 1 & -4 & 11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \qquad \boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Determine if $A\boldsymbol{x} = \boldsymbol{b}$ is consistent *without doing any row operations.*