Math 218D: Week 6 Discussion

Study Copy

October 5, 2023

Problem 1. To verify if $\boldsymbol{v} \in \operatorname{Col}(A)$ we must check the equation ____________________________.

Problem 2. Determine if $\boldsymbol{v} = \begin{bmatrix} 6 \\ 12 \\ 2 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \\ -3 & 9 & -12 \end{bmatrix}$.

Problem 3. To verify if $\boldsymbol{v} \in \operatorname{Span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ we must check ____.

Problem 4. Determine if $\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}^\intercal \in \operatorname{Span}\{\begin{bmatrix} 1 & -3 & -3 \end{bmatrix}^\intercal, \begin{bmatrix} -1 & 4 & 3 \end{bmatrix}^\intercal\}$.

Problem 5. Suppose that A is a 3×3 matrix satisfying the following three equations.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \qquad \operatorname{rref}\left[\begin{array}{ccc|c} & & & 1 \\ & A & & 1 \\ & & & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \operatorname{rref}\left[\begin{array}{ccc|c} & & & 1 \\ & A^\intercal & & 1 \\ & & & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & -11 \\ 0 & 1 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Note that $\operatorname{rref}(A)$ and $\operatorname{rref}(A^\intercal)$ can be inferred from the second and third equations above.

(*a*) The vector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\intercal$ belongs to exactly one of the four fundamental subspaces of A. Select this space.

○ The null space. ○ The row space. ○ The column space. ○ The left null space.

(*b*) The vector $\begin{bmatrix} 1 & -2 & -1 \end{bmatrix}^\intercal$ belongs to exactly one of the four fundamental subspaces of A. Select this space.

○ The null space. ○ The row space. ○ The column space. ○ The left null space.

(*c*) Determine if $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^\intercal \in \operatorname{Null}(A)$.

(*d*) Find a nonzero vector in the left null space of A.

Math 218D: Week 7 Discussion

Study Copy

October 12, 2023

Problem 1. By definition, what does it mean to call a list of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ *linearly dependent*?

Problem 2. By definition, what does it mean to call a list of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ *linearly independent*?

Problem 3. Determine if $\{\begin{bmatrix}1 & -3 & 1\end{bmatrix}^\intercal, \begin{bmatrix}-4 & 13 & -3\end{bmatrix}^\intercal, \begin{bmatrix}5 & -17 & 3\end{bmatrix}^\intercal\}$ is independent.

Problem 4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbb{R}^n$ and let A be an $m \times n$ matrix such that $\{A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3\}$ is linearly independent. Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

Problem 5. The columns of a matrix A are independent if and only if ____________________________.

Problem 6. Consider the calculations

$$\operatorname{rref}\overbrace{\begin{bmatrix} 9 & 4 & 4 \\ -36 & -16 & -16 \\ 20 & 9 & 4 \\ -49 & -22 & -12 \end{bmatrix}}^{A} = \begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & -44 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \operatorname{rref}\overbrace{\begin{bmatrix} 9 & -36 & 20 & -49 \\ 4 & -16 & 9 & -22 \\ 4 & -16 & 4 & -12 \end{bmatrix}}^{A^\intercal} = \begin{bmatrix} 1 & -4 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find the pivot solutions to $A\boldsymbol{v} = \boldsymbol{O}$. These vectors form a basis of ______________.

(b) Find the pivot solutions to $A^\intercal\boldsymbol{v} = \boldsymbol{O}$. These vectors form a basis of ______________.

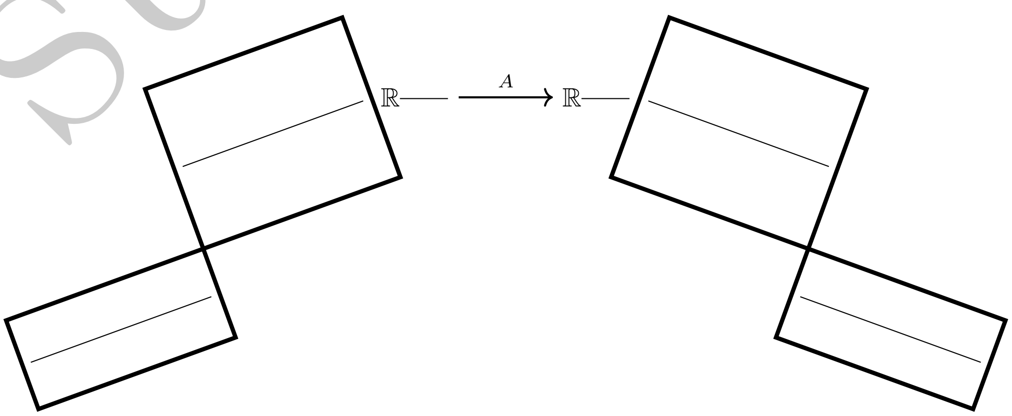
(c) Find the pivot columns of A. These vectors form a basis of ______________.

(d) Find the nonzero rows of $\operatorname{rref}(A)$. These vectors form a basis of ______________.

(e) The pivot columns of $A^\intercal$ form a basis of ______________.

(f) The nonzero rows of $\operatorname{rref}(A^\intercal)$ form a basis of ______________.

(g) Fill in the blanks in the figure below.



Math 218D: Week 8 Discussion

Study Copy

October 19, 2023

Problem 1. Suppose that A is a matrix satisfying

$$\operatorname{Col}(A^\intercal) = \operatorname{Span}\{\begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix}^\intercal, \begin{bmatrix}0 & 3 & 2 & 4\end{bmatrix}^\intercal\} \qquad \operatorname{Null}(A^\intercal) = \operatorname{Span}\{\begin{bmatrix}2 & 1 & 1\end{bmatrix}^\intercal\}$$

(a) Draw the picture of the four fundamental subspaces of A, including their dimensions

(b) Determine if $\boldsymbol{v} = \begin{bmatrix}1 & 0 & 2 & -1\end{bmatrix}^\intercal$ satisfies $A\boldsymbol{v} = \boldsymbol{O}$.

(c) Determine if $\boldsymbol{b} = \begin{bmatrix}3 & 5 & 2\end{bmatrix}^\intercal$ makes the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent.

(d) Explain why $\operatorname{Null}(A) \neq \operatorname{Span}\{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^\intercal, \begin{bmatrix}2 & 1 & 0 & 0\end{bmatrix}^\intercal\}$.

Problem 2. Suppose that an $n \times n$ matrix P is *symmetric* ($P^\intercal = P$) and *idempotent* ($P^2 = P$).

(a) Show that the matrix $I_n - P$ is symmetric.

(b) Show that the matrix $I_n - P$ is idempotent.

(c) Show that every vector $\boldsymbol{v} \in \mathbb{R}^n$ satisfies the equation

$$\langle P\boldsymbol{v}, (I_n - P)\boldsymbol{v} \rangle = 0$$

Hint. Start with the adjoint formula.

Problem 3. Let $\mathbb{1}_d$ be the vector in $\mathbb{R}^d$ whose every coordinate is equal to one (for example $\mathbb{1}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\intercal$ and $\mathbb{1}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\intercal$).

(a) Find the projection matrix P onto $V = \operatorname{Span}\{\mathbb{1}_3\}$.

(b) Find the projection of $\boldsymbol{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\intercal$ onto $V = \operatorname{Span}\{\mathbb{1}_3\}$.

(c) Describe what happens when we project $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\intercal$ onto $V = \operatorname{Span}\{\mathbb{1}_3\}$.