Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: Solutions

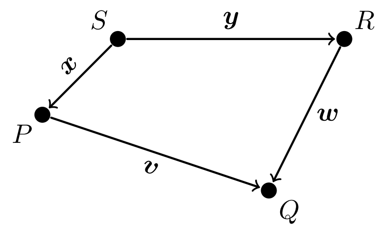
NetID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

September 29, 2023

- There are 100 points and 6 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.



Problem 1. This figure depicts displacement vectors $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ between points P, Q, R, and S in $\mathbb{R}^5$ (so each of P, Q, R, and S has five coordinates). Throughout this problem, assume that A is a matrix satisfying $A\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $A\boldsymbol{w} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

(3 pts) (*a*) Which point is the *tail* of $\boldsymbol{v}$? √ P ○ Q ○ R ○ S

(3 pts) (*b*) Only one of the following formulas for $\boldsymbol{w}$ is correct. Select this formula.
○ $\boldsymbol{w} = \overrightarrow{QR}$ ○ $\boldsymbol{w} = -\overrightarrow{RQ}$ ○ $\boldsymbol{w} = \vec{Q}$ √ $\boldsymbol{w} = \overrightarrow{RQ}$ ○ $\boldsymbol{w} = \vec{R}$

(3 pts) (*c*) Only one of the following formulas for $\overrightarrow{RP}$ is correct. Select this formula.
○ $\overrightarrow{RP} = \boldsymbol{x} + \boldsymbol{y}$ √ $\overrightarrow{RP} = \boldsymbol{x} - \boldsymbol{y}$ ○ $\overrightarrow{RP} = -\boldsymbol{x} - \boldsymbol{y}$ ○ $\overrightarrow{RP} = \boldsymbol{w} + \boldsymbol{v}$ ○ $\overrightarrow{RP} = \boldsymbol{y}$

(3 pts) (*d*) The number of rows of A is 2 and the number of columns of A is 5.

(4 pts) (*e*) If we assumed that P and Q have coordinates $P(1, 1, 1, 1, 1)$ and $Q(1, 1, 2, 1, 1)$, then only one of the following statements is guaranteed to be correct. Select this statement.
○ The second column of A is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ○ The second column of A is $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$ ○ The third column of A is $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$
○ The third column of A is $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$ √ The third column of A is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(8 pts) (*f*) Calculate the matrix-vector product $A(\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{w})$. Clearly explain your reasoning to receive credit.

Solution. The diagram demonstrates that $\boldsymbol{x} - \boldsymbol{y} = \overrightarrow{RP} = \boldsymbol{w} - \boldsymbol{v}$, which implies

$$A(\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{w}) = A(\boldsymbol{w} - \boldsymbol{v} + \boldsymbol{w}) = A(2 \cdot \boldsymbol{w} - \boldsymbol{v}) = 2 \cdot A\boldsymbol{w} - A\boldsymbol{v} = 2 \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \end{bmatrix}$$

Problem 2. The equation below depicts the result of multiplying the incidence matrix A of a digraph G by another matrix B.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \overset{B}{\begin{bmatrix}1&1&1&0&7&3\\3&0&1&1&5&9\\0&0&1&0&7&0\\2&1&1&0&7&6\\0&0&1&0&7&0\end{bmatrix}} = \overset{AB}{\begin{bmatrix}-2&0&-1&-1&-5&*\\0&0&0&0&0&*\\3&0&1&1&5&*\\0&0&1&0&7&*\\-1&0&-1&0&-7&*\end{bmatrix}}$$

Note that the entries in the last column of AB are unknown and marked as $*$.

(6 pts) (*a*) The number of nodes in G is __5__, the number of arrows in G is __5__, and $\chi(G) =$ __0__.

(5 pts) (*b*) Which of the following is the correct visualization of the second arrow a_2 in the digraph G?

✓ $v_1 \xrightarrow{a_2} v_3$ ○ $v_3 \xrightarrow{a_2} v_1$ ○ $v_2 \xrightarrow{a_2} v_3$

○ $v_1 \xrightarrow{a_2} v_2$ ○ $v_2 \xrightarrow{a_2} v_1$

(4 pts) (*c*) Suppose we make G a weighted digraph by assigning a weight of 5 to the second arrow a_2 and a weight of 7 to all of the other arrows. Then the net flow through the last node of G is __-7__.

(4 pts) (*d*) What is the missing column of AB?

○ $\begin{bmatrix}-2\\0\\3\\0\\-1\end{bmatrix}$ ○ $\begin{bmatrix}3\\9\\0\\6\\0\end{bmatrix}$ ✓ $\begin{bmatrix}-6\\0\\9\\0\\-3\end{bmatrix}$ ○ $\begin{bmatrix}1\\3\\0\\2\\0\end{bmatrix}$ ○ not enough information to tell

(5 pts) (*e*) Which of these statements best articulates the relationship of A to the terms "singular" and "nonsingular"?

✓ A is singular. ○ A is nonsingular. ○ A is neither singular nor nonsingular.

○ A is either singular or nonsingular, but we do not have enough information to decide which.

(12 pts) **Problem 3.** Select every matrix below whose rank is two (each option is worth 2pts).

✓ $\begin{bmatrix}0&0&8&1&47&73&80&42\\0&0&0&6&42&54&48&12\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}$ ○ $\begin{bmatrix}1&4&3&2&9\\1&4&3&2&9\\1&4&3&2&9\\1&4&3&2&9\\2&8&6&4&18\end{bmatrix}$ ✓ $\begin{bmatrix}1&-1\\4&9\\0&5\\-2&-2\\5&7\\-4&-4\\-1&3\\-2&-5\\0&3\end{bmatrix}$ ○ $\begin{bmatrix}0&1&0&5&7&0\\0&0&1&7&3&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}$

○ $\begin{bmatrix}2&2&3&4&2&2&5&65&76&1\\0&0&3&4&6&7&5&122&100&7\\0&0&0&7&4&1&8&86&71&1\\0&0&0&0&7&3&1&69&42&7\\0&0&0&0&0&9&2&75&40&1\\0&0&0&0&0&0&8&48&16&2\\0&0&0&0&0&0&0&0&0&7\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}$ ✓ $\begin{bmatrix}0&5&10&15&3\\0&0&0&0&3\\0&0&0&0&0\end{bmatrix}$

(12 pts) **Problem 4.** An $n \times n$ matrix M is called *idempotent* if $M^2 = M$.

Suppose that $A = XMX^{-1}$ where M is idempotent. Show that A is idempotent. **You must avoid circular logic to receive credit.**

Solution. We are given $A = XMX^{-1}$ where M is idempotent, which means that $M^2 = M$. We wish to demonstrate that A is idempotent, which means we want $A^2 = A$. This is then accomplished with

$$A^2 = (XMX^{-1})(XMX^{-1}) = XMX^{-1}XMX^{-1} = XMI_nMX^{-1} = XMMX^{-1} = XMX^{-1} = A$$

Problem 5. The data below depicts the result of representing the Gauß-Jordan algorithm applied to a matrix A with elementary matrices.

$$\overset{E_3}{\begin{bmatrix}1&0&0&0\\0&1&0&-7\\0&0&1&0\\0&0&0&1\end{bmatrix}}\overset{E_2}{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&0&0&1\end{bmatrix}}\begin{bmatrix}&&&\\&E_1&&\\&&&\end{bmatrix}\overset{A}{\begin{bmatrix}1&0&6&9&0&0\\0&-5&-10&-25&0&-35\\0&1&2&5&1&7\\0&0&0&0&0&1\end{bmatrix}}=\overset{R}{\begin{bmatrix}1&0&6&9&0&0\\0&1&2&5&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}}$$

Note that the elementary matrix E_1 is not explicitly described.

(4 pts) (a) The *second* elementary row operation used in the algorithm was ______ $\boldsymbol{r}_3 - \boldsymbol{r}_2 \to \boldsymbol{r}_3$ ______. **You must use the proper notation we learned in class to represent this elementary row operation.**

(8 pts) (b) Find E_1. Clearly explain your reasoning to receive credit.

Solution. This is the elementary matrix corresponding to the first elementary row operation in the Gauß-Jordan algorithm when applied to A, which is $-\frac{1}{5} \cdot \boldsymbol{r}_2 \to \boldsymbol{r}_2$. The elementary matrix we are looking for is obtained by applying this operation to the identity matrix, which gives

$$\overset{I_4}{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}\xrightarrow{-{}^1\!/\!_5\cdot\boldsymbol{r}_2\to\boldsymbol{r}_2}\overset{E_1}{\begin{bmatrix}1&0&0&0\\0&-{}^1\!/\!_5&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}$$

Problem 6. The data below depicts a $PA = LU$ factorization along with the inverse marix L^{-1} and a vector $\boldsymbol{b}$.

$$\overset{P}{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ & L & \\ & & \end{bmatrix} \overset{U}{\begin{bmatrix} 1 & 3 & 9 & 19 & 4 \\ 0 & 0 & 5 & 10 & 6 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \qquad L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ 1 & 2 & 1 & 3 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ h-1 \\ 0 \\ h+1 \end{bmatrix}$$

Note that A and L are not explicitly described, and that $\boldsymbol{b}$ is defined in terms of a constant h.

(4 pts) (a) $\operatorname{nullity}(U^\intercal) =$ ___1___

(12 pts) (b) There is only one value of h that makes the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent. Find this value of h. Clearly explain your reasoning to receive credit.

Solution. In class we demonstrated that the solutions to $A\boldsymbol{x} = \boldsymbol{b}$ are the solutions to $U\boldsymbol{x} = \boldsymbol{y}$ where $L\boldsymbol{y} = P\boldsymbol{b}$. Conveniently, we are given L^{-1}, so we can quickly solve $L\boldsymbol{y} = P\boldsymbol{b}$ with

$$\boldsymbol{y} = \overset{L^{-1}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ 1 & 2 & 1 & 3 \end{bmatrix}} \overset{P}{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \overset{\boldsymbol{b}}{\begin{bmatrix} 0 \\ h-1 \\ 0 \\ h+1 \end{bmatrix}} = \overset{L^{-1}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ 1 & 2 & 1 & 3 \end{bmatrix}} \overset{P\boldsymbol{b}}{\begin{bmatrix} h+1 \\ 0 \\ 0 \\ h-1 \end{bmatrix}} = \begin{bmatrix} h+1 \\ 3h+3 \\ 3h+3 \\ 4h-2 \end{bmatrix}$$

Now, in augmented form, the system $U\boldsymbol{x} = \boldsymbol{y}$ is

$$\left[\begin{array}{ccccc|c} 1 & 3 & 9 & 19 & 4 & h+1 \\ 0 & 0 & 5 & 10 & 6 & 3h+3 \\ 0 & 0 & 0 & 0 & 4 & 3h+3 \\ 0 & 0 & 0 & 0 & 0 & 4h-2 \end{array}\right]$$

The only way to avoid a pivot in the augmented column is to force $4h - 2 = 0$, which only works when $h = 1/2$.