Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: ______________________ *NetID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

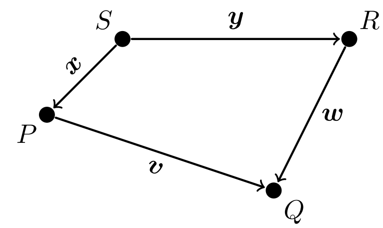
Signature: ______________________

September 29, 2023

- There are 100 points and 6 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

Problem 1. This figure depicts displacement vectors $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ between points P, Q, R, and S in $\mathbb{R}^5$ (so each of P, Q, R, and S has five coordinates). Throughout this problem, assume that A is a matrix satisfying $A\boldsymbol{v} = \begin{bmatrix}2\\3\end{bmatrix}$ and $A\boldsymbol{w} = \begin{bmatrix}7\\1\end{bmatrix}$.



(3 pts) (*a*) Which point is the *tail* of $\boldsymbol{v}$? ○ P ○ Q ○ R ○ S

(3 pts) (*b*) Only one of the following formulas for $\boldsymbol{w}$ is correct. Select this formula.
○ $\boldsymbol{w} = \overrightarrow{QR}$ ○ $\boldsymbol{w} = -\overrightarrow{RQ}$ ○ $\boldsymbol{w} = \vec{Q}$ ○ $\boldsymbol{w} = \overrightarrow{RQ}$ ○ $\boldsymbol{w} = \vec{R}$

(3 pts) (*c*) Only one of the following formulas for $\overrightarrow{RP}$ is correct. Select this formula.
○ $\overrightarrow{RP} = \boldsymbol{x} + \boldsymbol{y}$ ○ $\overrightarrow{RP} = \boldsymbol{x} - \boldsymbol{y}$ ○ $\overrightarrow{RP} = -\boldsymbol{x} - \boldsymbol{y}$ ○ $\overrightarrow{RP} = \boldsymbol{w} + \boldsymbol{v}$ ○ $\overrightarrow{RP} = \boldsymbol{y}$

(3 pts) (*d*) The number of rows of A is ________ and the number of columns of A is ________.

(4 pts) (*e*) If we assumed that P and Q have coordinates $P(1,1,1,1,1)$ and $Q(1,1,2,1,1)$, then only one of the following statements is guaranteed to be correct. Select this statement.
○ The second column of A is $\begin{bmatrix}2\\3\end{bmatrix}$ ○ The second column of A is $\begin{bmatrix}7\\1\end{bmatrix}$ ○ The third column of A is $\begin{bmatrix}-2\\-3\end{bmatrix}$
○ The third column of A is $\begin{bmatrix}7\\1\end{bmatrix}$ ○ The third column of A is $\begin{bmatrix}2\\3\end{bmatrix}$

(8 pts) (*f*) Calculate the matrix-vector product $A(\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{w})$. Clearly explain your reasoning to receive credit.

Problem 2. The equation below depicts the result of multiplying the incidence matrix A of a digraph G by another matrix B.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \overbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 7 & 3 \\ 3 & 0 & 1 & 1 & 5 & 9 \\ 0 & 0 & 1 & 0 & 7 & 0 \\ 2 & 1 & 1 & 0 & 7 & 6 \\ 0 & 0 & 1 & 0 & 7 & 0 \end{bmatrix}}^{B} = \overbrace{\begin{bmatrix} -2 & 0 & -1 & -1 & -5 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 3 & 0 & 1 & 1 & 5 & * \\ 0 & 0 & 1 & 0 & 7 & * \\ -1 & 0 & -1 & 0 & -7 & * \end{bmatrix}}^{AB}$$

Note that the entries in the last column of AB are unknown and marked as $*$.

(6 pts) (*a*) The number of nodes in G is _______, the number of arrows in G is _______, and $\chi(G) =$ _______.

(5 pts) (*b*) Which of the following is the correct visualization of the second arrow a_2 in the digraph G?

○ $v_1 \xrightarrow{a_2} v_3$ ○ $v_3 \xrightarrow{a_2} v_1$ ○ $v_2 \xrightarrow{a_2} v_3$

○ $v_1 \xrightarrow{a_2} v_2$ ○ $v_2 \xrightarrow{a_2} v_1$

(4 pts) (*c*) Suppose we make G a weighted digraph by assigning a weight of 5 to the second arrow a_2 and a weight of 7 to all of the other arrows. Then the net flow through the last node of G is _______.

(4 pts) (*d*) What is the missing column of AB?

○ $\begin{bmatrix} -2 \\ 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}$ ○ $\begin{bmatrix} 3 \\ 9 \\ 0 \\ 6 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} -6 \\ 0 \\ 9 \\ 0 \\ -3 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ ○ not enough information to tell

(5 pts) (*e*) Which of these statements best articulates the relationship of A to the terms "singular" and "nonsingular"?

○ A is singular. ○ A is nonsingular. ○ A is neither singular nor nonsingular.

○ A is either singular or nonsingular, but we do not have enough information to decide which.

(12 pts) **Problem 3.** Select every matrix below whose rank is two (each option is worth 2pts).

○ $\begin{bmatrix} 0 & 0 & 8 & 1 & 47 & 73 & 80 & 42 \\ 0 & 0 & 0 & 6 & 42 & 54 & 48 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 & 4 & 3 & 2 & 9 \\ 1 & 4 & 3 & 2 & 9 \\ 1 & 4 & 3 & 2 & 9 \\ 1 & 4 & 3 & 2 & 9 \\ 2 & 8 & 6 & 4 & 18 \end{bmatrix}$ ○ $\begin{bmatrix} 1 & -1 \\ 4 & 9 \\ 0 & 5 \\ -2 & -2 \\ 5 & 7 \\ -4 & -4 \\ -1 & 3 \\ -2 & -5 \\ 0 & 3 \end{bmatrix}$ ○ $\begin{bmatrix} 0 & 1 & 0 & 5 & 7 & 0 \\ 0 & 0 & 1 & 7 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

○ $\begin{bmatrix} 2 & 2 & 3 & 4 & 2 & 2 & 5 & 65 & 76 & 1 \\ 0 & 0 & 3 & 4 & 6 & 7 & 5 & 122 & 100 & 7 \\ 0 & 0 & 0 & 7 & 4 & 1 & 8 & 86 & 71 & 1 \\ 0 & 0 & 0 & 0 & 7 & 3 & 1 & 69 & 42 & 7 \\ 0 & 0 & 0 & 0 & 0 & 9 & 2 & 75 & 40 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 48 & 16 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 & 5 & 10 & 15 & 3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(12 pts) **Problem 4.** An $n \times n$ matrix M is called *idempotent* if $M^2 = M$.

Suppose that $A = XMX^{-1}$ where M is idempotent. Show that A is idempotent. **You must avoid circular logic to receive credit.**

Problem 5. The data below depicts the result of representing the Gauß-Jordan algorithm applied to a matrix A with elementary matrices.

$$\overset{E_3}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \overset{E_2}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} & & & \\ & E_1 & & \\ & & & \end{bmatrix} \overset{A}{\begin{bmatrix} 1 & 0 & 6 & 9 & 0 & 0 \\ 0 & -5 & -10 & -25 & 0 & -35 \\ 0 & 1 & 2 & 5 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & 0 & 6 & 9 & 0 & 0 \\ 0 & 1 & 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

Note that the elementary matrix E_1 is not explicitly described.

(4 pts) (a) The *second* elementary row operation used in the algorithm was ______________________________. **You must use the proper notation we learned in class to represent this elementary row operation.**

(8 pts) (b) Find E_1. Clearly explain your reasoning to receive credit.

Problem 6. The data below depicts a $PA = LU$ factorization along with the inverse marix L^{-1} and a vector $\boldsymbol{b}$.

$$\overset{P}{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ & L & \\ & & \end{bmatrix} \overset{U}{\begin{bmatrix} 1 & 3 & 9 & 19 & 4 \\ 0 & 0 & 5 & 10 & 6 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \qquad L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ 1 & 2 & 1 & 3 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ h-1 \\ 0 \\ h+1 \end{bmatrix}$$

Note that A and L are not explicitly described, and that $\boldsymbol{b}$ is defined in terms of a constant h.

(4 pts) (*a*) $\operatorname{nullity}(U^\intercal) =$ ________

(12 pts) (*b*) There is only one value of h that makes the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent. Find this value of h. Clearly explain your reasoning to receive credit.