Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: Solutions

NetID: ____________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________________

October 27, 2023

- There are 100 points and 6 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

Problem 1. Each of the three equations below provides information about the eigenvalues of A (the matrix in the first of the three equations).

$$\overset{A}{\begin{bmatrix} 2 & -3 & 5 & -6 \\ 1 & 0 & 3 & -8 \\ -1 & -3 & 8 & -6 \\ -1 & 1 & 0 & 6 \end{bmatrix}} \overset{\boldsymbol{v}_1}{\begin{bmatrix} 3 \\ 4 \\ 3 \\ -1 \end{bmatrix}} = \begin{bmatrix} 15 \\ 20 \\ 15 \\ -5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -5 & 6 \\ -1 & 3 & -3 & 8 \\ 1 & 3 & -5 & 6 \\ 1 & -1 & 0 & -3 \end{bmatrix} \overset{\boldsymbol{v}_2}{\begin{bmatrix} 2 \\ * \\ * \\ * \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \operatorname{rref} \begin{bmatrix} 2 & 3 & -5 & 6 \\ -1 & 4 & -3 & 8 \\ 1 & 3 & -4 & 6 \\ 1 & -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the vector $\boldsymbol{v}_2$ from the second equation is missing three entries marked $*$.

(4 pts) (*a*) The vector $\boldsymbol{v}_1$ is an eigenvector of A corresponding to the eigenvalue $\lambda =$ __5__.

(5 pts) (*b*) Only one of the following statements correctly characterizes the vector $\boldsymbol{v}_2$. Select this statement.

√ $\boldsymbol{v}_2 \in \mathcal{E}_A(3)$ ◯ $\boldsymbol{v}_2 \in \mathcal{E}_A(-3)$ ◯ $\boldsymbol{v}_2 \in \mathcal{E}_A(2)$ ◯ $\boldsymbol{v}_2 \in \mathcal{E}_A(-2)$ ◯ $\boldsymbol{v}_2 \notin \mathcal{E}_A(\lambda)$ for any value of λ

(5 pts) (*c*) The eigenvalue $\lambda =$ __4__ of A satisfies $\operatorname{gm}_A(\lambda) = 2$.

(4 pts) (*d*) Which of the following statements correctly summarizes the data of the eigenvalues of A?

◯ A has two eigenvalues with geometric multiplicity two.

√ A has two eigenvalues with geometric multiplicity one and one eigenvalue with geometric multiplicity two.

◯ A has one eigenvalue with geometric multiplicity one and two eigenvalues with geometric multiplicity two.

◯ A has three eigenvalues with geometric multiplicity two.

◯ A has three eigenvalues with geometric multiplicity one.

(10 pts) **Problem 2.** Suppose that G is a directed graph with exactly one connected component and that A is the incidence matrix of G. The equation below depicts the result of Brian's attempt to multiply A by a matrix X and record the result as Y.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \overset{X}{\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}} = \overset{Y}{\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ -3 & 0 & -2 & -2 \\ 0 & 0 & -1 & -2 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix}}$$

Despite his best efforts, Brian made a mistake in his calculation! One (and only one) of the columns of Y above is incorrect. Identify which column of Y was calculated incorrectly. Clearly explain your reasoning to receive credit.

Solution. First, note that A must be 7×6 because X is 6×4 and Y is 7×4. The Euler characteristic formula from class then allows us to infer $h_1(G)$.

$$\overset{7-6=1}{\chi(G)} = \overset{1}{h_0(G)} - \underset{0}{h_1(G)}$$

So, $h_1(G) = 0$. But then $\dim \operatorname{Null}(A) = h_1(G) = 0$, which means that $\operatorname{Null}(A)$ is home only to the zero vector. The problem is that the second column of Y is the zero vector, but the second column of X is not the zero vector!

(12 pts) **Problem 3.** Select every matrix below whose columns are linearly independent (each option is worth 2pts).

○ $\begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ √ $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 0 \\ 4 & 4 \end{bmatrix}$ ○ $\begin{bmatrix} 3 & 1 & 3 & 3 & 3 & 1 & 2 & 4 & 2 \\ 2 & 3 & 2 & 1 & 1 & 2 & 4 & 2 & 4 \\ 4 & 1 & 3 & 3 & 1 & 4 & 1 & 1 & 4 \\ 3 & 4 & 4 & 4 & 2 & 1 & 3 & 2 & 3 \\ 1 & 4 & 1 & 3 & 3 & 4 & 4 & 2 & 4 \end{bmatrix}$ ○ $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 1 & 3 & 2 \\ 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix}$

○ $\begin{bmatrix} 4 & 24 & 9 & 37 & 6 & 7 & 131 & 4 & 4 & 6 \\ 0 & 0 & 3 & 3 & 9 & 5 & 86 & 4 & 1 & 8 \\ 0 & 0 & 0 & 0 & 6 & 1 & 43 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 8 & 8 & 8 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ √ $\begin{bmatrix} 1 & 1 & 6 & 3 \\ 0 & 5 & 8 & 9 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(12 pts) **Problem 4.** Suppose that A is a 9×7 matrix and that $\boldsymbol{v} \in \mathbb{R}^7$ and $\boldsymbol{w} \in \mathbb{R}^9$ satisfy the two equations

$$A\boldsymbol{v} = 9 \cdot \boldsymbol{w} \qquad\qquad A^\intercal \boldsymbol{w} = 5 \cdot \boldsymbol{v}$$

where $\boldsymbol{v} \neq \boldsymbol{O}$. Show that $\boldsymbol{v}$ is an eigenvector of the Gramian of A and identify the corresponding eigenvalue.

Solution. We want to demonstrate that $A^\intercal A\boldsymbol{v} = \lambda \cdot \boldsymbol{v}$ for some identifiable λ. To do so, note that

$$A^\intercal A\boldsymbol{v} = A^\intercal(9 \cdot \boldsymbol{w}) = 9 \cdot A^\intercal \boldsymbol{w} = 9 \cdot 5 \cdot \boldsymbol{v} = 45 \cdot \boldsymbol{v}$$

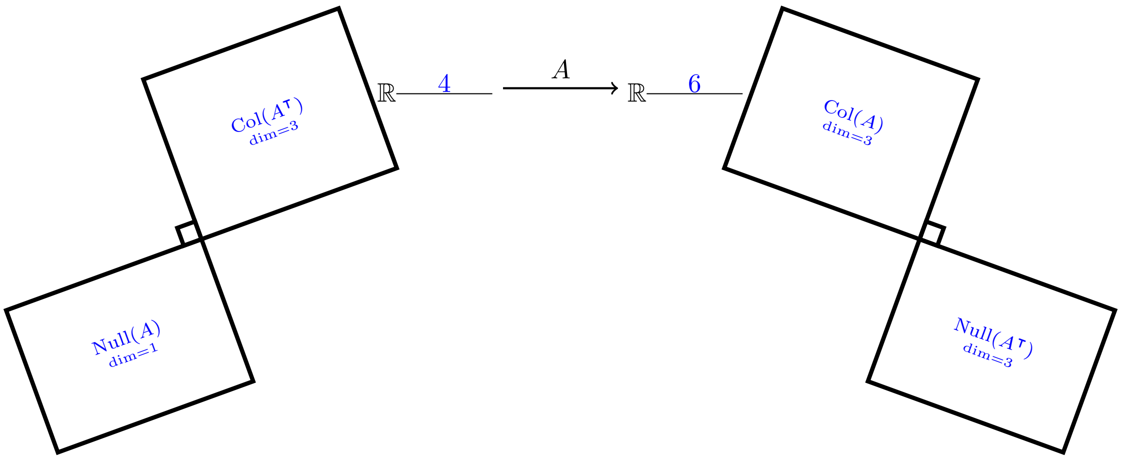
This demonstrates that $\boldsymbol{v}$ is an eigenvector of $A^\intercal A$ with corresponding eigenvalue $\lambda = 45$.

Problem 5. Suppose that A is a matrix satisfying the following properties

$$\begin{bmatrix}1\\2\\0\\1\end{bmatrix}, \begin{bmatrix}2\\h\\0\\2\end{bmatrix} \in \operatorname{Null}(A) \qquad \operatorname{Null}(A^\intercal) = \operatorname{Span}\left\{\begin{bmatrix}5\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}9\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}3\\8\\2\\0\\0\\0\end{bmatrix}\right\}$$

Note that the second vector in the first equation above is defined in terms of a constant labeled h.

(10 pts) (*a*) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(4 pts) (*b*) The only valid value of h is $h =$ ___4___.

(4 pts) (*c*) Only one of the following formulas for $\boldsymbol{b}$ makes the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent. Select this vector.

○ $\boldsymbol{b} = \begin{bmatrix}1\\0\\0\\1\\0\\0\end{bmatrix}$ ○ $\boldsymbol{b} = \begin{bmatrix}0\\1\\0\\0\\1\\0\end{bmatrix}$ ○ $\boldsymbol{b} = \begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix}$ √ $\boldsymbol{b} = \begin{bmatrix}0\\0\\0\\1\\1\\1\end{bmatrix}$ ○ $\boldsymbol{b} = \begin{bmatrix}1\\1\\1\\1\\0\\1\end{bmatrix}$

(8 pts) (*d*) Find the projection matrix onto Null(A).

Solution. The digram tells us that $\dim \operatorname{Null}(A) = 1$ and we know $\boldsymbol{v}_1 = \begin{bmatrix}1 & 2 & 0 & 1\end{bmatrix}^\intercal \in \operatorname{Null}(A)$, so the one-dimensional projection formula gives

$$P_{\operatorname{Null}(A)} = \frac{1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1\boldsymbol{v}_1^\intercal = \frac{1}{6}\begin{bmatrix}1\\2\\0\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 0 & 1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1 & 2 & 0 & 1\\2 & 4 & 0 & 2\\0 & 0 & 0 & 0\\1 & 2 & 0 & 1\end{bmatrix}$$

Problem 6. The data below depicts a matrix A and the result of multiplying the projection matrix onto $\operatorname{Col}(A)$ by another matrix Y.

$$A = \begin{bmatrix} * & 1 & 3 \\ * & * & 0 \\ 2 & 1 & -1 \\ 1 & * & -1 \end{bmatrix} \qquad \left[\quad P_{\operatorname{Col}(A)} \quad \right] \overbrace{\begin{bmatrix} -3 & 24 & 3 & 5 & 4 \\ 7 & -3 & -29 & 0 & -1 \\ 7 & 16 & -29 & 1 & -2 \\ -10 & 2 & -20 & -1 & 3 \end{bmatrix}}^{Y} = \begin{bmatrix} 0 & 23 & 4 & 5 & 3 \\ 4 & -2 & -30 & 0 & 0 \\ 4 & 17 & -30 & 1 & -1 \\ 2 & -2 & -16 & -1 & -1 \end{bmatrix}$$

Note that A is missing several entries marked $*$.

(6 pts) (a) Only one of the columns of Y is in the column space of A. Select this column.

○ first column ○ second column ○ third column ✓ fourth column ○ fifth column

(9 pts) (b) Let $\boldsymbol{b} = \begin{bmatrix} 4 & -1 & -2 & 3 \end{bmatrix}^\intercal$ (the last column of Y). Find the least squares approximate solution $\widehat{\boldsymbol{x}}$ to $A\boldsymbol{x} = \boldsymbol{b}$.

Solution. We have two options for solving the least squares problem

$$A\widehat{\boldsymbol{x}} = P\boldsymbol{b} \qquad\qquad A^\intercal A\widehat{x} = A^\intercal \boldsymbol{b}$$

The second option is significantly less attractive than the first because of all of the unknown entries in A. The data gives us $P\boldsymbol{b}$, so we can at least look at the system $A\widehat{\boldsymbol{x}} = P\boldsymbol{b}$, which is

$$\overbrace{\begin{bmatrix} * & 1 & 3 \\ * & * & 0 \\ 2 & 1 & -1 \\ 1 & * & -1 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{bmatrix}}^{\widehat{\boldsymbol{x}}} = \overbrace{\begin{bmatrix} 3 \\ 0 \\ -1 \\ -1 \end{bmatrix}}^{P\boldsymbol{b}}$$

We see here that $P\boldsymbol{b}$ is the last column of A! This means that $\widehat{x} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\intercal$.

(7 pts) (c) Let $\boldsymbol{b} = \begin{bmatrix} 24 & -3 & 16 & 2 \end{bmatrix}^\intercal$ (the second column of Y) and let $\widehat{x}$ be the least squares approximate solution to $A\boldsymbol{x} = \boldsymbol{b}$. Find the error E in this approximation.

Solution. This is

$$E = \|\boldsymbol{b} - A\widehat{x}\|^2 = \|\boldsymbol{b} - P\boldsymbol{b}\|^2 = \left\| \begin{bmatrix} 24 \\ -3 \\ 16 \\ 2 \end{bmatrix} - \begin{bmatrix} 23 \\ -2 \\ 17 \\ -2 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} 1 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\|^2 = 19$$