Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: Solutions

NetID:

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

December 1, 2023

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Suppose that $A = QR$ where A, Q, and R are given by

$$A = \begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix} \qquad Q = \frac{1}{h}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -x \\ 1 & x & 1 \\ x & -1 & -1 \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{5} & 4 & 0 \\ 0 & 2 & 7 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$$

Note that the columns of A have been labeled $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ and that the formula for Q depends on variables x and h.

(6 pts) (*a*) $\operatorname{rank}(A) =$ __3__, $\operatorname{rank}(R) =$ __3__, and $\operatorname{rank}(Q) =$ __3__

(5 pts) (*b*) $h =$ __$\sqrt{3+x^2}$__ (your formula for h here should depend on the variable x)

(8 pts) (*c*) $\det(R) =$ __10__, $\det(RQ^\intercal Q) =$ __10__, and $\det(RA^\intercal A) =$ __1000__

(6 pts) (*d*) If $\boldsymbol{q}_2$ is the second column of Q, then $\langle \boldsymbol{q}_2, \boldsymbol{a}_1 \rangle =$ __0__, $\langle \boldsymbol{q}_2, \boldsymbol{a}_2 \rangle =$ __2__, and $\langle \boldsymbol{q}_2, \boldsymbol{a}_3 \rangle =$ __7__

(6 pts) (*e*) If $\boldsymbol{q}_1$ is the first column of Q, then only one of the following statements is correct. Select this statement.

○ $\operatorname{proj}_{\boldsymbol{q}_1}(\boldsymbol{a}_1) = \boldsymbol{O}$ ○ $\operatorname{proj}_{\boldsymbol{q}_1}(\boldsymbol{a}_2) = \boldsymbol{O}$ √ $\operatorname{proj}_{\boldsymbol{q}_1}(\boldsymbol{a}_3) = \boldsymbol{O}$ ○ none of these equations is correct

(10 pts) (*f*) Find the projection of $\boldsymbol{b} = \begin{bmatrix} h^2 & 0 & 0 & 0 \end{bmatrix}^\intercal$ onto $\operatorname{Col}(Q)$ (your answer will deend on the variable x).

Solution. According to our formulas from class, this is

$$\begin{aligned} P\boldsymbol{b} &= \overbrace{\frac{1}{h}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -x \\ 1 & x & 1 \\ x & -1 & -1 \end{bmatrix}}^{Q} \overbrace{\frac{1}{h}\begin{bmatrix} 1 & -1 & 1 & x \\ 1 & 1 & x & -1 \\ -1 & -x & 1 & -1 \end{bmatrix}}^{Q^\intercal} \overbrace{\begin{bmatrix} h^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}^{\boldsymbol{b}} \\ &= \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -x \\ 1 & x & 1 \\ x & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ x \\ x \\ x \end{bmatrix} \end{aligned}$$

Problem 2. The following equation depicts $A = XBX^{-1}$, which tells us that A is *similar* to B.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \overset{X}{\begin{bmatrix} i & 1 & 2 & -1 \\ 1 & i & -1 & -i \\ 1 & -3 & 0 & -1 \\ 1 & -1 & i & 2 \end{bmatrix}} \overset{B}{\begin{bmatrix} 5 & 2 & 0 & 9 \\ 0 & 7 & i & 4 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & i \end{bmatrix}} \begin{bmatrix} & & \\ & X^{-1} & \\ & & \end{bmatrix}$$

Note that several entries in X and in B are nonreal complex numbers and that B is upper triangular.

(4 pts) (*a*) $\operatorname{trace}(A) =$ $\underline{\quad 13+i \quad}$ and $\det(A) =$ $\underline{\quad 35\,i \quad}$

(4 pts) (*b*) If $\boldsymbol{x}_1$ is the first column of X and $\boldsymbol{x}_2$ is the second column of X, then $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle =$ $\underline{\quad -4 \quad}$.

(8 pts) (*c*) Note that $\operatorname{trace}(X) = 2i+2$. This calculation allows us to decide whether or not each of the following statements is true. Select each true statement (each option is worth 2pts).

- ◯ $\lambda = 2i+2$ is an eigenvalue of X
- √ X has at least one nonreal eigenvalue
- ◯ the coefficient of t^3 in $\chi_X(t)$ is $2i+2$
- √ X cannot be similar to any Hermitian matrix

(3 pts) (*d*) The algebraic multiplicity of every eigenvalue λ of A is $\operatorname{am}_A(\lambda) =$ $\underline{\quad 1 \quad}$.

(10 pts) (*e*) Note that $\lambda = 5$ and $\lambda = 7$ are both eigenvalues of A. Find bases of $\mathcal{E}_A(5)$ and $\mathcal{E}_A(7)$ and determine if $\mathcal{E}_A(5) \perp \mathcal{E}_A(7)$.

Hint. Start by finding bases of $\mathcal{E}_B(5)$ and $\mathcal{E}_B(7)$. How do bases of these eigenspaces then translate into bases of $\mathcal{E}_A(5)$ and $\mathcal{E}_A(7)$?

Solution. Note that

$$\mathcal{E}_B(5) = \operatorname{Null} \overset{5\cdot I_4 - B}{\begin{bmatrix} 0 & -2 & 0 & -9 \\ 0 & -2 & -i & -4 \\ 0 & 0 & 4 & -i \\ 0 & 0 & 0 & -i+5 \end{bmatrix}} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} \quad \mathcal{E}_B(7) = \operatorname{Null} \overset{7\cdot I_4 - B}{\begin{bmatrix} 2 & -2 & 0 & -9 \\ 0 & 0 & -i & -4 \\ 0 & 0 & 6 & -i \\ 0 & 0 & 0 & -i+7 \end{bmatrix}} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

The key statement from class is then that $\mathcal{E}_A(\lambda) = X \cdot \mathcal{E}_B(\lambda)$, which translates as

$$\mathcal{E}_A(5) = \operatorname{Span}\left\{\begin{bmatrix} & & \\ & X & \\ & & \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\} \qquad \mathcal{E}_A(7) = \operatorname{Span}\left\{\begin{bmatrix} & & \\ & X & \\ & & \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} i+1 \\ i+1 \\ -2 \\ 0 \end{bmatrix}\right\}$$

The issue of whether or not $\mathcal{E}_A(5) \perp \mathcal{E}_A(7)$ is then resolved with a single inner product.

$$\langle \begin{bmatrix} i & 1 & 1 & 1 \end{bmatrix}^\intercal, \begin{bmatrix} i+1 & i+1 & -2 & 0 \end{bmatrix}^\intercal \rangle = 0$$

It turns out that $\mathcal{E}_A(5)$ and $\mathcal{E}_A(7)$ are indeed orthogonal!

Problem 3. The data below depicts an invertible real-symmetric matrix S, an invertible matrix T, and the characteristic polynomial $\chi_S(t)$ of S (which has been partially factored).

$$S = \begin{bmatrix} 2 & -1 & 1 & 2 \\ -1 & 2 & -1 & -2 \\ 1 & -1 & 2 & 2 \\ 2 & -2 & 2 & 5 \end{bmatrix} \qquad T = \begin{bmatrix} -7 & 1 & -1 & -1 \\ 0 & 1 & 2 & -1 \\ -10 & 14 & 1 & -2 \\ 1 & 5 & -2 & 1 \end{bmatrix} \qquad \chi_S(t) = \left(t^2 - 2\,t + 1\right)\left(t^2 - 9\,t + 8\right)$$

Throughout this problem, let $A = M^{-1}T$ where $M = S^{-1}T$.

(6 pts) (*a*) Determine the definiteness of S. Clearly explain your reasoning to receive credit.

Solution. We are given $\chi_S(t)$ as the product of quadratics, which can be further factored as

$$\chi_S(t) = (t-8)(t-1)(t-1)(t-1) = (t-1)^3(t-8)$$

This tells us that E-Vals$(S) = \{1, 8\}$. All eigenvalues of S are positive, so S is *positive definite*.

(10 pts) (*b*) Show that A is similar to S.

Hint. This can be done purely with symbols.

Solution. We are given that $A = M^{-1}T$ where $M = S^{-1}T$. We wish to demonstrate that $A = XSX^{-1}$ for some X. To do so, note that

$$A = M^{-1}T = (S^{-1}T)^{-1}T = T^{-1}ST$$

This is $A = XSX^{-1}$ with $X = T^{-1}$, which demonstrates that A is indeed similar to S.

(14 pts) **Problem 4.** Suppose that $\boldsymbol{u}(t)$ is the solution to $\boldsymbol{u}' = A\boldsymbol{u}$ with $\boldsymbol{u}(0) = \boldsymbol{u}_0$ where

$$A = \begin{bmatrix} -1 & 1 \\ 0 & a \end{bmatrix} \qquad \boldsymbol{u}_0 = (a+1)\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Note that the matrix A and the vector $\boldsymbol{u}_0$ are defined in terms of a real variable a which is known to satisfy $a \neq -1$. The two coordintes u_1 and u_2 of $\boldsymbol{u}(t)$ depend both on t and a and can thus be interpreted as scalar fields. Calculate the partial derivatives $\dfrac{\partial u_1}{\partial a}$ and $\dfrac{\partial u_2}{\partial a}$.

Solution. The relevant formula from class is $\boldsymbol{u}(t) = \exp(At)\boldsymbol{u}_0$. We hope that A diagonalizes as $A = XDX^{-1}$ so that we could write $\boldsymbol{u}(t) = X\exp(Dt)X^{-1}\boldsymbol{u}_0$.

The good news is that A is upper triangular, so immediately we find E-Vals$(A) = \{-1, a\}$. The eigenspaces are

$$\mathcal{E}_A(-1) = \operatorname{Null}\overset{-I_2-A}{\begin{bmatrix} 0 & -1 \\ 0 & -a-1 \end{bmatrix}} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \qquad \mathcal{E}_A(a) = \operatorname{Null}\overset{a\cdot I_2-A}{\begin{bmatrix} a+1 & -1 \\ 0 & 0 \end{bmatrix}} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ a+1 \end{bmatrix}\right\}$$

This gives the diagonalization

$$\overset{A}{\begin{bmatrix} -1 & 1 \\ 0 & a \end{bmatrix}} = \overset{X}{\begin{bmatrix} 1 & 1 \\ 0 & a+1 \end{bmatrix}}\overset{D}{\begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix}}\overset{X^{-1}}{\frac{1}{a+1}\begin{bmatrix} a+1 & -1 \\ 0 & 1 \end{bmatrix}}$$

The solution $\boldsymbol{u}(t)$ to the initial value problem is then

$$\begin{aligned}
\boldsymbol{u}(t) &= \overset{X}{\begin{bmatrix} 1 & 1 \\ 0 & a+1 \end{bmatrix}}\overset{\exp(Dt)}{\begin{bmatrix} e^{-t} & \\ & e^{at} \end{bmatrix}}\overset{X^{-1}}{\frac{1}{a+1}\begin{bmatrix} a+1 & -1 \\ 0 & 1 \end{bmatrix}}\overset{\boldsymbol{u}_0}{(a+1)\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix}} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & a+1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{at} \end{bmatrix}\begin{bmatrix} 3\,a+2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & a+1 \end{bmatrix}\begin{bmatrix} (3\,a+2)e^{-t} \\ e^{at} \end{bmatrix} \\
&= \begin{bmatrix} (3\,a+2)e^{-t} + e^{at} \\ (a+1)e^{at} \end{bmatrix}
\end{aligned}$$

We now have the formulas

$$u_1 = (3\,a+2)e^{-t} + e^{at} \qquad u_2 = (a+1)e^{at}$$

Our partial derivatives are then

$$\frac{\partial u_1}{\partial a} = 3\,e^{-t} + te^{at} \qquad \frac{\partial u_2}{\partial a} = e^{at} + t\,(a+1)e^{at}$$