Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: ______________________ *NetID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

December 1, 2023

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Suppose that $A = QR$ where A, Q, and R are given by

$$A = \begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix} \qquad Q = \frac{1}{h}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -x \\ 1 & x & 1 \\ x & -1 & -1 \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{5} & 4 & 0 \\ 0 & 2 & 7 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$$

Note that the columns of A have been labeled $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ and that the formula for Q depends on variables x and h.

(6 pts) (*a*) $\operatorname{rank}(A) =$ ________, $\operatorname{rank}(R) =$ ________, and $\operatorname{rank}(Q) =$ ________

(5 pts) (*b*) $h =$ ________________________ (your formula for h here should depend on the variable x)

(8 pts) (*c*) $\det(R) =$ ________________, $\det(RQ^{\intercal}Q) =$ ________________, and $\det(RA^{\intercal}A) =$ ________________

(6 pts) (*d*) If $\boldsymbol{q}_2$ is the second column of Q, then $\langle \boldsymbol{q}_2, \boldsymbol{a}_1 \rangle =$ ________, $\langle \boldsymbol{q}_2, \boldsymbol{a}_2 \rangle =$ ________, and $\langle \boldsymbol{q}_2, \boldsymbol{a}_3 \rangle =$ ________

(6 pts) (*e*) If $\boldsymbol{q}_1$ is the first column of Q, then only one of the following statements is correct. Select this statement.

○ $\operatorname{proj}_{\boldsymbol{q}_1}(\boldsymbol{a}_1) = \boldsymbol{O}$ ○ $\operatorname{proj}_{\boldsymbol{q}_1}(\boldsymbol{a}_2) = \boldsymbol{O}$ ○ $\operatorname{proj}_{\boldsymbol{q}_1}(\boldsymbol{a}_3) = \boldsymbol{O}$ ○ none of these equations is correct

(10 pts) (*f*) Find the projection of $\boldsymbol{b} = \begin{bmatrix} h^2 & 0 & 0 & 0 \end{bmatrix}^{\intercal}$ onto $\operatorname{Col}(Q)$ (your answer will deend on the variable x).

Problem 2. The following equation depicts $A = XBX^{-1}$, which tells us that A is *similar* to B.

$$\left[\begin{array}{ccc} & & \\ & A & \\ & & \end{array}\right] = \overset{X}{\begin{bmatrix} i & 1 & 2 & -1 \\ 1 & i & -1 & -i \\ 1 & -3 & 0 & -1 \\ 1 & -1 & i & 2 \end{bmatrix}} \overset{B}{\begin{bmatrix} 5 & 2 & 0 & 9 \\ 0 & 7 & i & 4 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & i \end{bmatrix}} \left[\begin{array}{ccc} & & \\ & X^{-1} & \\ & & \end{array}\right]$$

Note that several entries in X and in B are nonreal complex numbers and that B is upper triangular.

(4 pts) (*a*) $\operatorname{trace}(A) =$ ______________ and $\det(A) =$ ______________

(4 pts) (*b*) If $\boldsymbol{x}_1$ is the first column of X and $\boldsymbol{x}_2$ is the second column of X, then $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle =$ ______________.

(8 pts) (*c*) Note that $\operatorname{trace}(X) = 2i+2$. This calculation allows us to decide whether or not each of the following statements is true. Select each true statement (each option is worth 2pts).

- ○ $\lambda = 2i + 2$ is an eigenvalue of X
- ○ X has at least one nonreal eigenvalue
- ○ the coefficient of t^3 in $\chi_X(t)$ is $2i + 2$
- ○ X cannot be similar to any Hermitian matrix

(3 pts) (*d*) The algebraic multiplicity of every eigenvalue λ of A is $\operatorname{am}_A(\lambda) =$ _______.

(10 pts) (*e*) Note that $\lambda = 5$ and $\lambda = 7$ are both eigenvalues of A. Find bases of $\mathcal{E}_A(5)$ and $\mathcal{E}_A(7)$ and determine if $\mathcal{E}_A(5) \perp \mathcal{E}_A(7)$.
Hint. Start by finding bases of $\mathcal{E}_B(5)$ and $\mathcal{E}_B(7)$. How do bases of these eigenspaces then translate into bases of $\mathcal{E}_A(5)$ and $\mathcal{E}_A(7)$?

Problem 3. The data below depicts an invertible real-symmetric matrix S, an invertible matrix T, and the characteristic polynomial $\chi_S(t)$ of S (which has been partially factored).

$$S = \begin{bmatrix} 2 & -1 & 1 & 2 \\ -1 & 2 & -1 & -2 \\ 1 & -1 & 2 & 2 \\ 2 & -2 & 2 & 5 \end{bmatrix} \qquad T = \begin{bmatrix} -7 & 1 & -1 & -1 \\ 0 & 1 & 2 & -1 \\ -10 & 14 & 1 & -2 \\ 1 & 5 & -2 & 1 \end{bmatrix} \qquad \chi_S(t) = \left(t^2 - 2\,t + 1\right)\left(t^2 - 9\,t + 8\right)$$

Throughout this problem, let $A = M^{-1}T$ where $M = S^{-1}T$.

(6 pts) (a) Determine the definiteness of S. Clearly explain your reasoning to receive credit.

(10 pts) (b) Show that A is similar to S.

Hint. This can be done purely with symbols.

(14 pts) **Problem 4.** Suppose that $\boldsymbol{u}(t)$ is the solution to $\boldsymbol{u}' = A\boldsymbol{u}$ with $\boldsymbol{u}(0) = \boldsymbol{u}_0$ where

$$A = \begin{bmatrix} -1 & 1 \\ 0 & a \end{bmatrix} \qquad\qquad \boldsymbol{u}_0 = (a+1) \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Note that the matrix A and the vector $\boldsymbol{u}_0$ are defined in terms of a real variable a which is known to satisfy $a \neq -1$. The two coordintes u_1 and u_2 of $\boldsymbol{u}(t)$ depend both on t and a and can thus be interpreted as scalar fields. Calculate the partial derivatives $\dfrac{\partial u_1}{\partial a}$ and $\dfrac{\partial u_2}{\partial a}$.