Linear Approximations

Problem 1. Let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{f}(x, y) = \begin{bmatrix} x^2 + \sin(x) \\ x(y-2) \\ -3\,xy + y^2 \end{bmatrix}$$

Use the local linearization of $\boldsymbol{f}$ at the point $P = (0, 1)$ to approximate $f\left(1/2, 1/2\right)$.

Problem 2. Let $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

Use the local linearization of $\boldsymbol{r}$ at $t = 1/2\,\pi$ to approximate $\boldsymbol{r}(1/2\,\pi + 1/3)$.

Problem 3. Consider $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{F}(r, \theta, z) = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ z \end{bmatrix}$$

Show that $\det(D\boldsymbol{F}) = r$.

Problem 4. Consider $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{F}(\rho, \varphi, \theta) = \begin{bmatrix} \rho\cos(\theta)\sin(\varphi) \\ \rho\sin(\theta)\sin(\varphi) \\ \rho\cos(\varphi) \end{bmatrix}$$

Show that $\det(D\boldsymbol{F}) = \rho^2 \sin(\varphi)$.