Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: Solutions

Unique ID:

I have adhered to the Duke Community Standard in completing this exam.

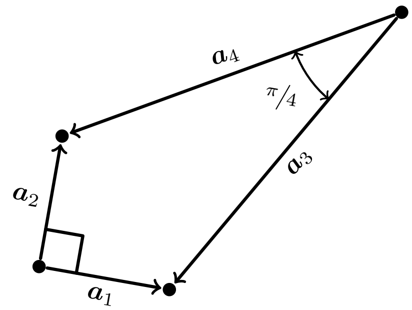
Signature:

September 27, 2024

- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. The matrix A below is 5×4 whose columns are labeled as $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_4$. The figure below depicts a geometric visualization of the columns of A.

$$A = \begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \\ | & | & | & | \end{bmatrix}$$



Note that the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ form a right angle and that $\boldsymbol{a}_3$ and $\boldsymbol{a}_4$ form an angle of $\pi/4$. Additionally, it is known that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are unit vectors and that $\|\boldsymbol{a}_3\| = \|\boldsymbol{a}_4\| = \dfrac{\sqrt{3}+3}{2}$.

(2 pts) (*a*) Each of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_4$ is a vector in $\mathbb{R}^*$ where $* =$ __5__.

(2 pts) (*b*) The expression $A\boldsymbol{v}$ is a vector with __5__ coordinates, provided that the vector $\boldsymbol{v}$ has __4__ coordinates.

(5 pts) (*c*) Only one of the following equations is correct. Select this equation.

○ $\begin{bmatrix} A \end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$ ○ $\begin{bmatrix} A \end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$ √ $\begin{bmatrix} A \end{bmatrix}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$ ○ $\begin{bmatrix} A \end{bmatrix}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$

(5 pts) (*d*) Only one of the following scalar quantities is *negative*. Select this quantity.

○ $\langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle$ ○ $\boldsymbol{a}_2^\intercal \boldsymbol{a}_1$ √ $\boldsymbol{a}_4^\intercal(-\boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{a}_4)$ ○ The $(3,4)$ entry of $A^\intercal A$.

Solution. The first two are $\langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle = 0$ and $\boldsymbol{a}_2^\intercal \boldsymbol{a}_1 = \langle \boldsymbol{a}_2, \boldsymbol{a}_1 \rangle = 0$ because $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are orthogonal. The $(3,4)$ entry of $A^\intercal A$ is $\langle \boldsymbol{a}_3, \boldsymbol{a}_4 \rangle > 0$ because $\boldsymbol{a}_3$ and $\boldsymbol{a}_4$ are *acute*. The third is $\boldsymbol{a}_4^\intercal(-\boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{a}_4) = \langle \boldsymbol{a}_4, -\boldsymbol{a}_3 \rangle = -\langle \boldsymbol{a}_4, \boldsymbol{a}_3 \rangle < 0$.

(8 pts) (*e*) Fill-in the blanks in this equation with valid integers: $\operatorname{trace}(A^\intercal A) =$ __8__ $+$ __3__ $\cdot \sqrt{3}$. Show your work below to receive credit.

Solution. The trace of $A^\intercal A$ is the sum of the diagonal entries of $A^\intercal A$. We know from class that

$$A^\intercal A = \begin{bmatrix} \langle \boldsymbol{a}_1, \boldsymbol{a}_1 \rangle & \langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle & \langle \boldsymbol{a}_1, \boldsymbol{a}_3 \rangle & \langle \boldsymbol{a}_1, \boldsymbol{a}_4 \rangle \\ \langle \boldsymbol{a}_2, \boldsymbol{a}_1 \rangle & \langle \boldsymbol{a}_2, \boldsymbol{a}_2 \rangle & \langle \boldsymbol{a}_2, \boldsymbol{a}_3 \rangle & \langle \boldsymbol{a}_2, \boldsymbol{a}_4 \rangle \\ \langle \boldsymbol{a}_3, \boldsymbol{a}_1 \rangle & \langle \boldsymbol{a}_3, \boldsymbol{a}_2 \rangle & \langle \boldsymbol{a}_3, \boldsymbol{a}_3 \rangle & \langle \boldsymbol{a}_3, \boldsymbol{a}_4 \rangle \\ \langle \boldsymbol{a}_4, \boldsymbol{a}_1 \rangle & \langle \boldsymbol{a}_4, \boldsymbol{a}_2 \rangle & \langle \boldsymbol{a}_4, \boldsymbol{a}_3 \rangle & \langle \boldsymbol{a}_4, \boldsymbol{a}_4 \rangle \end{bmatrix} = \begin{bmatrix} \|\boldsymbol{a}_1\|^2 & * & * & * \\ * & \|\boldsymbol{a}_2\|^2 & * & * \\ * & * & \|\boldsymbol{a}_3\|^2 & * \\ * & * & * & \|\boldsymbol{a}_4\|^2 \end{bmatrix}$$

All of the entries marked $*$ are irrelevant to our calculation (even though we could quickly infer the $(1,2)$, $(2,1)$, $(3,4)$, and $(4,3)$ entries). Our desired quantity is then

$$\begin{aligned} \operatorname{trace}(A^\intercal A) &= 1^2 + 1^2 + \left(\frac{\sqrt{3}+3}{2}\right)^2 + \left(\frac{\sqrt{3}+3}{2}\right)^2 \\ &= 2 + \frac{2}{2^2}(\sqrt{3}+3)^2 \\ &= 2 + \frac{1}{2}((\sqrt{3})^2 + 3^2 + 2 \cdot 3 \cdot \sqrt{3}) \\ &= 2 + \frac{1}{2}(3 + 9 + 6\sqrt{3}) \\ &= 8 + 3 \cdot \sqrt{3} \end{aligned}$$

Problem 2. Each of the matrices A, B, and C below is a "full rank" matrix.

$$A = \begin{bmatrix} -388 & 119 & -438 & 414 & 14 \\ 291 & -53 & 235 & -152 & -359 \\ -213 & 712 & -365 & 725 & 300 \end{bmatrix} \quad B = \begin{bmatrix} 423 & 423 & 423 & 423 & 423 \\ 417 & 417 & 416 & 416 & 416 \\ 432 & 431 & 431 & 431 & 431 \\ 449 & 449 & 449 & 448 & 448 \\ 379 & 379 & 379 & 379 & 378 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Note that the matrix C is the incidence of a directed graph G.

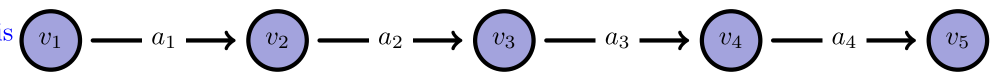
(3 pts) (a) $\operatorname{rank}(A) =$ __3__, $\operatorname{rank}(B) =$ __5__, and $\operatorname{rank}(C) =$ __4__

(6 pts) (b) $\operatorname{nullity}(A) =$ __2__, $\operatorname{nullity}(A^\intercal) =$ __0__, and $\operatorname{nullity}(A^\intercal A) =$ __2__

(9 pts) (c) Which (if any) of the following matrices is *invertible*? Select all that apply (1.5pts each).

○ A √ B ○ C ○ $A^\intercal A$ √ $B^\intercal B$ √ $C^\intercal C$

(6 pts) (d) The number of connected components of G is __1__, the circuit rank of G is __0__, and $\chi(G) =$ __1__.

Solution. The picture here is 

(10 pts) (e) Calculate the matrix $M = ABC$. This can be done quickly without complicated arithmetic, so no partial credit will be awarded for arithmetic errors. You can ensure an award of partial credit by correctly conveying the size of M and correctly aritculating any properties of matrix arithmetic that help calculate M.

Solution. First, A is 3×5, B is 5×5, and C is 5×4. This means that $M = ABC$ is a valid expression and expected to be a 3×4 matrix.

Of course, we cannot help but notice that the numbers in A and B are terrible. However, the numbers in C are not so bad. In fact, we also notice that the numbers in B are "close together".

The *associative property* of matrix multiplication asserts that $M = (AB)C = A(BC)$. It looks clear here that the calculation $M = A(BC)$ is preferred over $M = (AB)C$. This gives

$$\begin{aligned} M &= \begin{bmatrix} & A & \end{bmatrix} \left(\overset{B}{\begin{bmatrix} 423 & 423 & 423 & 423 & 423 \\ 417 & 417 & 416 & 416 & 416 \\ 432 & 431 & 431 & 431 & 431 \\ 449 & 449 & 449 & 448 & 448 \\ 379 & 379 & 379 & 379 & 378 \end{bmatrix}} \overset{C}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}} \right) \\ &= \overset{A}{\begin{bmatrix} -388 & 119 & -438 & 414 & 14 \\ 291 & -53 & 235 & -152 & -359 \\ -213 & 712 & -365 & 725 & 300 \end{bmatrix}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -438 & 119 & 414 & 14 \\ 235 & -53 & -152 & -359 \\ -365 & 712 & 725 & 300 \end{bmatrix} \end{aligned}$$

The key to this problem is that BC turns out to have a single 1 in every column and the rest of its entries are zero. Calculating $M = A(BC)$ is then selects columns three, two, four and five of A respectively.

Problem 3. Consider the following $EA = R$ factorization

$$\overset{E}{\begin{bmatrix} 8 & 1 & -3 & 0 & 7 \\ 7 & 1 & -4 & -2 & 5 \\ 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 4 & -1 & -2 & -9 \\ -1 & -4 & 1 & 2 & 9 \\ 2 & 8 & 0 & 6 & 4 \\ -1 & -4 & -1 & -8 & -13 \\ 0 & 0 & 1 & 5 & 11 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & 4 & 0 & 3 & 2 \\ 0 & 0 & 1 & 5 & 11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

(6 pts) (*a*) $\operatorname{rank}(E) =$ ___5___, $\operatorname{rank}(A) =$ ___2___, and $\operatorname{rank}(R) =$ ___2___.

(4 pts) (*b*) The columns of A satisfy all of the following equations. However, according to our terminology from class, only one of the following equations is called a *column relation*. Select this equation.

○ $\text{Col}_5 = -\text{Col}_1 + 6\ \text{Col}_3 + \text{Col}_4$ ✓ $\text{Col}_4 = 3\ \text{Col}_1 + 5\ \text{Col}_3$ ○ $\text{Col}_4 = 7\ \text{Col}_1 - \text{Col}_2 + 5\ \text{Col}_3$

○ $\text{Col}_4 = -\text{Col}_1 + \text{Col}_2 + 5\ \text{Col}_3$ ○ $\text{Col}_5 = 3\ \text{Col}_1 - \text{Col}_2 + 6\ \text{Col}_3 + \text{Col}_4$

(4 pts) (*c*) Only one of the following formulas for $\boldsymbol{b}$ makes the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent. Select this formula.

○ $\boldsymbol{b} = \begin{bmatrix} 1 & 2 & -3 & 0 & 0 \end{bmatrix}^\intercal$ ○ $\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^\intercal$

○ $\boldsymbol{b} = \begin{bmatrix} 0 & 1 & -2 & 1 & 0 \end{bmatrix}^\intercal$ ✓ $\boldsymbol{b} = \begin{bmatrix} 0 & 0 & -2 & 2 & -1 \end{bmatrix}^\intercal$

(10 pts) (*d*) If we use the Gauß-Jordan algorithm as articulated in class to define E as the product of elementary matrices, then it would take *eight* elementary matrices to do so $E = E_8E_7E_6E_5E_4E_3E_2E_1$. Find the elementary row operation that would define E_4 in this context and write this operation with proper notation in this blank: ___$\boldsymbol{r}_2 \leftrightarrow \boldsymbol{r}_3$___. ***Clearly label your row operations to receive full credit.***

Solution. We need to follow the Gauß-Jordan algorithm to row-reduce A, but we can stop after four operations. Here, we have

$$\overset{A}{\begin{bmatrix} 1 & 4 & -1 & -2 & -9 \\ -1 & -4 & 1 & 2 & 9 \\ 2 & 8 & 0 & 6 & 4 \\ -1 & -4 & -1 & -8 & -13 \\ 0 & 0 & 1 & 5 & 11 \end{bmatrix}} \xrightarrow{\substack{\boldsymbol{r}_2 + \boldsymbol{r}_1 \to \boldsymbol{r}_2 \\ \boldsymbol{r}_3 - 2\boldsymbol{r}_1 \to \boldsymbol{r}_3 \\ \boldsymbol{r}_4 + \boldsymbol{r}_1 \to \boldsymbol{r}_4}} \begin{bmatrix} 1 & 4 & -1 & -2 & -9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 10 & 22 \\ 0 & 0 & -2 & -10 & -22 \\ 0 & 0 & 1 & 5 & 11 \end{bmatrix} \xrightarrow{\boldsymbol{r}_2 \leftrightarrow \boldsymbol{r}_3} \begin{bmatrix} 1 & 4 & -1 & -2 & -9 \\ 0 & 0 & 2 & 10 & 22 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -10 & -22 \\ 0 & 0 & 1 & 5 & 11 \end{bmatrix}$$

The fourth elementary row operation is $\boldsymbol{r}_2 \leftrightarrow \boldsymbol{r}_3$.

(10 pts) **Problem 4.** Let A be a nonzero 2024×2024 matrix satisfying $A^2 = A$ and let $M = I_{2024} + A$. There is only one scalar value of c for which $M^{-1} = I_{2024} + c \cdot A$. Find this value of c and record your answer in this blank: $c =$ $\underline{\quad -1/2 \quad}$. ***You must clearly explain your work and avoid circular reasoning to receive credit.***

Solution. The purpose of M^{-1} is that it is the only matrix satisfying $MM^{-1} = I_{2024}$. So, we want

$$I_{2024} = (I + A)(I + c \cdot A) = I + c \cdot A + A + c \cdot A^2 = I + c \cdot A + A + c \cdot A = I + (2\,c + 1) \cdot A$$

The only way to get this to work is if $2\,c + 1 = 0$, or $c = -1/2$.

(10 pts) **Problem 5.** Consider P, L, and U below along with the calculation of $\operatorname{rref}[L \mid P\boldsymbol{b}]$ for $\boldsymbol{b} = \begin{bmatrix} 6 & 10 & 9 & 23 & 8 \end{bmatrix}^\intercal$.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 2 & 6 & 2 & 0 \\ 7 & 4 & 3 & 2 \\ 9 & 5 & 4 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & -4 & 2 & 0 & -8 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \operatorname{rref}\left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 6 \\ 4 & 5 & 0 & 0 & 23 \\ 2 & 6 & 2 & 0 & 10 \\ 7 & 4 & 3 & 2 & 8 \\ 9 & 5 & 4 & 1 & 9 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let A be the matrix satisfying $PA = LU$. The system $A\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions. Find the solution $\boldsymbol{x}$ obtained by setting every free variable in this system equal to one. Clearly explain your reasoning to receive credit.

Solution. Given $PA = LU$, the system $A\boldsymbol{x} = \boldsymbol{b}$ is solved by first solving $L\boldsymbol{y} = P\boldsymbol{b}$ for $\boldsymbol{y}$ and then solving $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$. We are given $\operatorname{rref}[L \mid P\boldsymbol{b}]$, which automatically tells us $\boldsymbol{y} = \begin{bmatrix} 2 & 3 & -6 & 0 \end{bmatrix}^\intercal$ in $L\boldsymbol{y} = P\boldsymbol{b}$.

Now, we turn our attention to the system $U\boldsymbol{x} = \boldsymbol{y}$. There are five variables in this system x_1, x_2, x_3, x_4, x_5. The pivots in U are in columns one, three, and four. This means that the dependent variables are x_1, x_3, x_4 and the free variables are x_2, x_5.

Normally, we would solve our system by relabeling the free variables as $x_2 = c_1$ and $x_5 = c_2$. The problem instructs us, however, to use $x_2 = x_5 = 1$. The system $U\boldsymbol{x} = \boldsymbol{y}$ is then

$$\begin{array}{rcrcrcrcrcrcrcccr} 2\,x_1 & - & 4\,(1) & + & 2\,x_3 & & & - & 8\,(1) & = & 2 & \to & x_1 & = & 13 \\ & & & & x_3 & + & 2\,x_4 & + & 3\,(1) & = & 3 & \to & x_3 & = & -6 \\ & & & & & - & 3\,x_4 & + & 3\,(1) & = & -6 & \to & x_4 & = & 3 \\ & & & & & & & & 0 & = & 0 & & & & \end{array}$$

The back-substution gives our desired solution as $\boldsymbol{x} = \begin{bmatrix} 13 & 1 & -6 & 3 & 1 \end{bmatrix}^\intercal$.