Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: ______________________ *Unique ID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

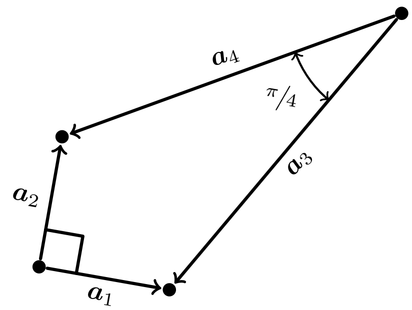
September 27, 2024

- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

Problem 1. The matrix A below is 5×4 whose columns are labeled as $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_4$. The figure below depicts a geometric visualization of the columns of A.

$$A = \begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \\ | & | & | & | \end{bmatrix}$$



Note that the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ form a right angle and that $\boldsymbol{a}_3$ and $\boldsymbol{a}_4$ form an angle of $\pi/4$. Additionally, it is known that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are unit vectors and that $\|\boldsymbol{a}_3\| = \|\boldsymbol{a}_4\| = \dfrac{\sqrt{3}+3}{2}$.

(2 pts) (a) Each of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_4$ is a vector in $\mathbb{R}^*$ where $* =$ _______.

(2 pts) (b) The expression $A\boldsymbol{v}$ is a vector with _______ coordinates, provided that the vector $\boldsymbol{v}$ has _______ coordinates.

(5 pts) (c) Only one of the following equations is correct. Select this equation.

○ $\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(5 pts) (d) Only one of the following scalar quantities is *negative*. Select this quantity.

○ $\langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle$ ○ $\boldsymbol{a}_2^\intercal \boldsymbol{a}_1$ ○ $\boldsymbol{a}_4^\intercal(-\boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{a}_4)$ ○ The $(3,4)$ entry of $A^\intercal A$.

(8 pts) (e) Fill-in the blanks in this equation with valid integers: $\operatorname{trace}(A^\intercal A) =$ _______ $+$ _______ $\cdot \sqrt{3}$. Show your work below to receive credit.

Problem 2. Each of the matrices A, B, and C below is a "full rank" matrix.

$$A = \begin{bmatrix} -388 & 119 & -438 & 414 & 14 \\ 291 & -53 & 235 & -152 & -359 \\ -213 & 712 & -365 & 725 & 300 \end{bmatrix} \quad B = \begin{bmatrix} 423 & 423 & 423 & 423 & 423 \\ 417 & 417 & 416 & 416 & 416 \\ 432 & 431 & 431 & 431 & 431 \\ 449 & 449 & 449 & 448 & 448 \\ 379 & 379 & 379 & 379 & 378 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Note that the matrix C is the incidence of a directed graph G.

(3 pts) (a) $\operatorname{rank}(A) =$ _______, $\operatorname{rank}(B) =$ _______, and $\operatorname{rank}(C) =$ _______

(6 pts) (b) $\operatorname{nullity}(A) =$ _______, $\operatorname{nullity}(A^\intercal) =$ _______, and $\operatorname{nullity}(A^\intercal A) =$ _______

(9 pts) (c) Which (if any) of the following matrices is *invertible*? Select all that apply (1.5pts each).

○ A ○ B ○ C ○ $A^\intercal A$ ○ $B^\intercal B$ ○ $C^\intercal C$

(6 pts) (d) The number of connected components of G is _______, the circuit rank of G is _______, and $\chi(G) =$ _______.

(10 pts) (e) Calculate the matrix $M = ABC$. This can be done quickly without complicated arithmetic, so no partial credit will be awarded for arithmetic errors. You can ensure an award of partial credit by correctly conveying the size of M and correctly aritculating any properties of matrix arithmetic that help calculate M.

Problem 3. Consider the following $EA = R$ factorization

$$\overset{E}{\begin{bmatrix} 8 & 1 & -3 & 0 & 7 \\ 7 & 1 & -4 & -2 & 5 \\ 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 4 & -1 & -2 & -9 \\ -1 & -4 & 1 & 2 & 9 \\ 2 & 8 & 0 & 6 & 4 \\ -1 & -4 & -1 & -8 & -13 \\ 0 & 0 & 1 & 5 & 11 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & 4 & 0 & 3 & 2 \\ 0 & 0 & 1 & 5 & 11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

(6 pts) (*a*) $\operatorname{rank}(E) =$ ________, $\operatorname{rank}(A) =$ ________, and $\operatorname{rank}(R) =$ ________.

(4 pts) (*b*) The columns of A satisfy all of the following equations. However, according to our terminology from class, only one of the following equations is called a *column relation*. Select this equation.

○ $\text{Col}_5 = -\text{Col}_1 + 6\,\text{Col}_3 + \text{Col}_4$ ○ $\text{Col}_4 = 3\,\text{Col}_1 + 5\,\text{Col}_3$ ○ $\text{Col}_4 = 7\,\text{Col}_1 - \text{Col}_2 + 5\,\text{Col}_3$

○ $\text{Col}_4 = -\text{Col}_1 + \text{Col}_2 + 5\,\text{Col}_3$ ○ $\text{Col}_5 = 3\,\text{Col}_1 - \text{Col}_2 + 6\,\text{Col}_3 + \text{Col}_4$

(4 pts) (*c*) Only one of the following formulas for $\boldsymbol{b}$ makes the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent. Select this formula.

○ $\boldsymbol{b} = \begin{bmatrix} 1 & 2 & -3 & 0 & 0 \end{bmatrix}^\intercal$ ○ $\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^\intercal$

○ $\boldsymbol{b} = \begin{bmatrix} 0 & 1 & -2 & 1 & 0 \end{bmatrix}^\intercal$ ○ $\boldsymbol{b} = \begin{bmatrix} 0 & 0 & -2 & 2 & -1 \end{bmatrix}^\intercal$

(10 pts) (*d*) If we use the Gauß-Jordan algorithm as articulated in class to define E as the product of elementary matrices, then it would take *eight* elementary matrices to do so $E = E_8E_7E_6E_5E_4E_3E_2E_1$. Find the elementary row operation that would define E_4 in this context and write this operation with proper notation in this blank: ______________________________. ***Clearly label your row operations to receive full credit.***

(10 pts) **Problem 4.** Let A be a nonzero 2024×2024 matrix satisfying $A^2 = A$ and let $M = I_{2024} + A$. There is only one scalar value of c for which $M^{-1} = I_{2024} + c \cdot A$. Find this value of c and record your answer in this blank: $c =$ __________. ***You must clearly explain your work and avoid circular reasoning to receive credit.***

(10 pts) **Problem 5.** Consider P, L, and U below along with the calculation of $\operatorname{rref}[L \mid P\boldsymbol{b}]$ for $\boldsymbol{b} = \begin{bmatrix}6 & 10 & 9 & 23 & 8\end{bmatrix}^{\intercal}$.

$$P = \begin{bmatrix}1&0&0&0&0\\0&0&0&1&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&1&0&0\end{bmatrix} \quad L = \begin{bmatrix}3&0&0&0\\4&5&0&0\\2&6&2&0\\7&4&3&2\\9&5&4&1\end{bmatrix} \quad U = \begin{bmatrix}2&-4&2&0&-8\\0&0&1&2&3\\0&0&0&-3&3\\0&0&0&0&0\end{bmatrix} \quad \operatorname{rref}\left[\begin{array}{cccc|c}3&0&0&0&6\\4&5&0&0&23\\2&6&2&0&10\\7&4&3&2&8\\9&5&4&1&9\end{array}\right] = \left[\begin{array}{cccc|c}1&0&0&0&2\\0&1&0&0&3\\0&0&1&0&-6\\0&0&0&1&0\\0&0&0&0&0\end{array}\right]$$

Let A be the matrix satisfying $PA = LU$. The system $A\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions. Find the solution $\boldsymbol{x}$ obtained by setting every free variable in this system equal to one. Clearly explain your reasoning to receive credit.