# Duke University

## Math 218D-2

### Matrices and Vectors

# Exam II

*Name:* Solutions

*Unique ID:* ____________________

*I have adhered to the Duke Community Standard in completing this exam.*

Signature: ________________________________

October 25, 2024

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

**Problem 1.** Suppose that $A\boldsymbol{x} = \boldsymbol{b}$ is a *consistent system* where $A$ is $31 \times 17$ and that $\boldsymbol{y}$ and $\boldsymbol{z}$ are solutions to this system where $\boldsymbol{y} \neq \boldsymbol{z}$ (this means that $\boldsymbol{y}$ and $\boldsymbol{z}$ are vectors satisfying $A\boldsymbol{y} = \boldsymbol{b}$ and $A\boldsymbol{z} = \boldsymbol{b}$).

(3 pts) (*a*) The vector $\boldsymbol{b}$ has __17__ coordinates and the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ have __31__ coordinates.

(4 pts) (*b*) Only one of the following statements is *guaranteed* to be correct. Select this statement.

○ $\boldsymbol{b}$ belongs to the row space of $A$ ○ $\boldsymbol{b}$ belongs to the null space of $A$

√ $\boldsymbol{b}$ belongs to the column space of $A$ ○ $\boldsymbol{b}$ belongs to the left null space of $A$

○ we do not have enough information to determine if $\boldsymbol{b}$ belongs to one of the four fundamental subspaces of $A$

(4 pts) (*c*) Only one of the following statements is *guaranteed* to accurately characterize $A^\intercal A\boldsymbol{y}$. Select this statement.

√ $A^\intercal A\boldsymbol{y}$ is a vector in the row space of $A$ ○ $A^\intercal A\boldsymbol{y}$ is a vector in the null space of $A$

○ $A^\intercal A\boldsymbol{y}$ is a vector in the column space of $A$ ○ $A^\intercal A\boldsymbol{y}$ is a vector in the left null space of $A$

(8 pts) (*d*) Prove that $\boldsymbol{y} - \boldsymbol{z} \in \operatorname{Null}(A^\intercal A)$. Use the space outside the box below to brainstorm your thoughts, but your proof must be neatly and succinctly presented in the box below. **You must avoid circular reasoning to receive credit.**

**Solution.** We know that $A\boldsymbol{y} = \boldsymbol{b}$ and that $A\boldsymbol{z} = \boldsymbol{b}$. We wish to prove that $\boldsymbol{y} - \boldsymbol{z} \in \operatorname{Null}(A^\intercal A)$, which is the same as demonstrating that $A^\intercal A(\boldsymbol{y} - \boldsymbol{z}) = \boldsymbol{O}$. The proof then succinctly fits into our box below:

$$A^\intercal A(\boldsymbol{y} - \boldsymbol{z}) = A^\intercal A\boldsymbol{y} - A^\intercal A\boldsymbol{z} = A^\intercal\boldsymbol{b} - A^\intercal\boldsymbol{b} = \boldsymbol{O}$$

(9 pts) (*e*) The facts that $\boldsymbol{y} \neq \boldsymbol{z}$ and that $\boldsymbol{y} - \boldsymbol{z} \in \operatorname{Null}(A^\intercal A)$ imply that some, but not all, of the following statements are *guaranteed* to be true. Select the true statements (1.5pts each).

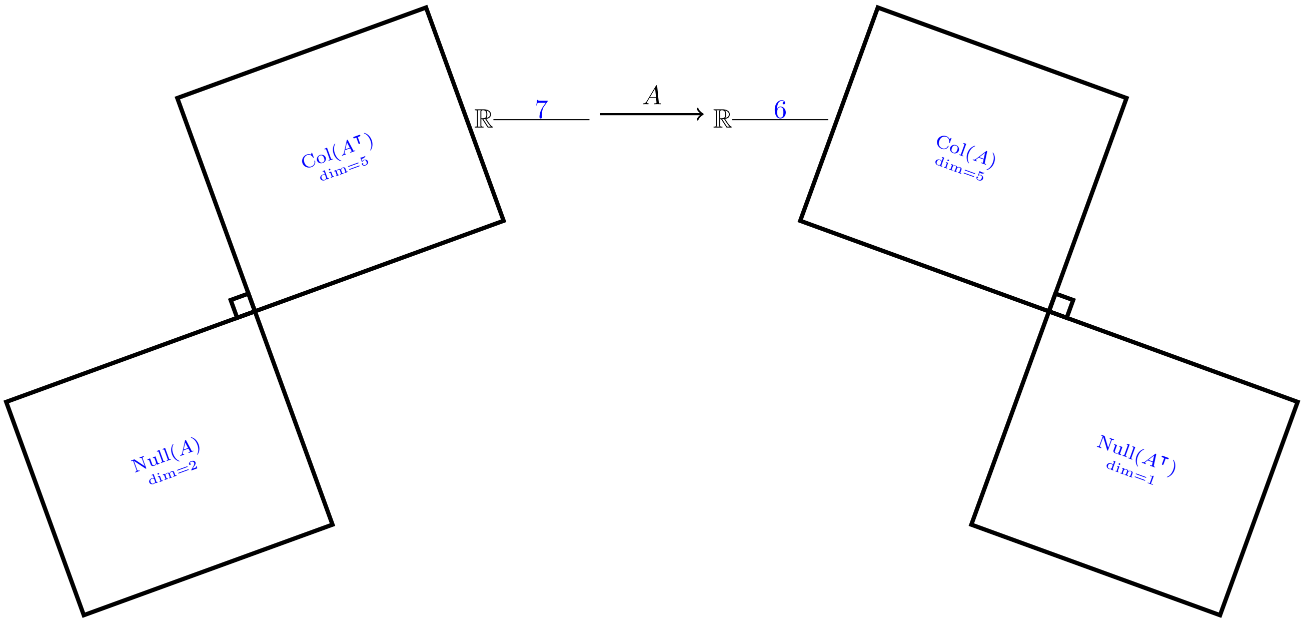
√ $\dim \operatorname{Null}(A^\intercal A) > 0$ ○ $\dim \operatorname{Null}(A^\intercal A) = 1$ ○ $A^\intercal A$ has independent columns

√ $A^\intercal A$ is singular ○ $\lambda = 0$ is *not* and eigenvalue of $A$ √ $\boldsymbol{y} - \boldsymbol{z}$ is an eigenvector of $A^\intercal A$

**Solution.** The fact that $\boldsymbol{y} \neq \boldsymbol{z}$ means that $\boldsymbol{y} - \boldsymbol{z} \neq \boldsymbol{O}$. The fact that $\boldsymbol{y} - \boldsymbol{z} \in \operatorname{Null}(A^\intercal A)$ then tells us that $\operatorname{Null}(A^\intercal A)$ contains a vector other than the zero vector. From this we conclude that $\dim \operatorname{Null}(A^\intercal A) > 0$ but we don't have enough information to say that $\dim \operatorname{Null}(A^\intercal A) = 1$. Since $\operatorname{nullity}(A^\intercal A) = \dim \operatorname{Null}(A^\intercal A) > 0$, we further conclude that $A^\intercal A$ has *dependent* columns and is therefore *singular* (so $\lambda = 0$ *is* and eigenvalue of $A^\intercal A$). Since $A^\intercal A(\boldsymbol{y} - \boldsymbol{z}) = \boldsymbol{O} = 0 \cdot (\boldsymbol{y} - \boldsymbol{z})$, we know that $\boldsymbol{y} - \boldsymbol{z}$ *is* an eigenvector of $A^\intercal A$ (with eigenvalue zero).

**Problem 2.** Suppose $A$ satisfies $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\operatorname{rref}(A^\intercal) = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(10 pts) ($a$) Fill in every missing label in the picture of the four fundamental subspaces of $A$ below, including the dimension of each fundamental subspace.




(3 pts) ($b$) Which of the following is the most accurate geometric description of the *left null space* of $A$?

○ a plane in $\mathbb{R}^6$ ○ a point with six coordinates ○ a line in $\mathbb{R}^7$ √ a line in $\mathbb{R}^6$ ○ a plane in $\mathbb{R}^7$

(4 pts) ($c$) Only one of the following statements accurately describes the columns of $A$. Select this statement.

○ $A$ has independent columns ○ every choice of five columns of $A$ will be independent

√ it is impossible to find six independent vectors among the columns of $A$

○ the first four columns of $A$ form a basis of $\operatorname{Col}(A)$

(4 pts) ($d$) The projection matrix $P$ onto the row space of $A$ is $P = X(X^\intercal X)^{-1}X^\intercal$ where $X$ is 7 × 5.

(10 pts) ($e$) Let $V$ be the vector space spanned by the rows of $\operatorname{rref}(A^\intercal)$. Find a basis of $V^\perp$. Clearly explain your work to receive credit.

**Solution.** From class we know that the nonzero rows of $\operatorname{rref}(A^\intercal)$ form a basis of $\operatorname{Col}(A)$. Including the rows of zeros doesn't change the span, so $V = \operatorname{Col}(A)$, which means we want to find a basis of $V^\perp = \operatorname{Null}(A^\intercal)$. The picture tells us that $\dim \operatorname{Null}(A^\intercal) = 1$, so we need only find one nonzero solution to $A^\intercal \boldsymbol{x} = \boldsymbol{O}$. The given $\operatorname{rref}(A^\intercal)$ tells us that the equations defining $A^\intercal \boldsymbol{x} = \boldsymbol{O}$ are

$$x_1 + 4x_2 = 0 \qquad x_3 = 0 \qquad x_4 = 0 \qquad x_5 = 0 \qquad x_6 = 0$$

The free variable is $x_2 = c_1$ so the general solution is

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^\intercal = \begin{bmatrix} -4c_1 & c_1 & 0 & 0 & 0 & 0 \end{bmatrix}^\intercal = c_1 \cdot \begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\intercal$$

This tells us that any nonzero multiple of $\begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\intercal$ is a valid basis of $V^\perp$.

**Problem 3.** One of the eigenvalues of $A = \begin{bmatrix} 5 & -5 & 0 & -5 \\ 25 & 0 & 25 & -5 \\ -5 & 5 & 0 & 5 \\ -25 & 0 & -25 & 5 \end{bmatrix}$ is $\lambda = 5$.

(10 pts) (*a*) Find a basis of $\mathcal{E}_A(5)$. Clearly explain your work to receive credit.

**Solution.** By definition, we want a basis of is

$$\mathcal{E}_A(5) = \operatorname{Null} \overbrace{\begin{bmatrix} 0 & 5 & 0 & 5 \\ -25 & 5 & -25 & 5 \\ 5 & -5 & 5 & -5 \\ 25 & 0 & 25 & 0 \end{bmatrix}}^{5\cdot I_4 - A} = \operatorname{Null} \overbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}^{\operatorname{rref}(5\cdot I_4 - A)} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

The calculation of $\operatorname{rref}(5 \cdot I_4 - A)$ is done where without row-reductions by observing the column relations $\operatorname{Col}_3 = \operatorname{Col}_1$ and $\operatorname{Col}_4 = \operatorname{Col}_2$.

(9 pts) (*b*) Let $P$ be the projection matrix onto $\mathcal{E}_A(5)$ and suppose that $\boldsymbol{v} \in \mathcal{E}_A(5)$. Some, but not all, of the following statements are correct. Select each correct statement (1.5pts each).

√ $A\boldsymbol{v} = 5 \cdot \boldsymbol{v}$ ○ $P\boldsymbol{v} = 5 \cdot \boldsymbol{v}$ ○ $A\boldsymbol{v} = \boldsymbol{O}$ √ $P\boldsymbol{v} = \boldsymbol{v}$ √ $\operatorname{trace}(P) = \operatorname{gm}_A(5)$ ○ $\operatorname{trace}(P) = 5$

(10 pts) (*c*) Let $\boldsymbol{b} = \begin{bmatrix} 4 & 0 & 6 & 2 \end{bmatrix}^\intercal$. If we assemble the basis vectors found for $\mathcal{E}_A(5)$ in part (*a*) of this problem into the columns of a matrix $B$, then we would find that $B(B^\intercal B)^{-1} B^\intercal \boldsymbol{b} = \begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix}^\intercal$. Use this information to calculate the error in the least squares problem associated to the system $B\boldsymbol{x} = \boldsymbol{b}$ and fill in the blank: $E =$ 52 . Clearly explain your work to receive credit.

**Solution.** The error in the least squares problem assiciated to $B\boldsymbol{x} = \boldsymbol{b}$ is the square length of the difference between $\boldsymbol{b}$ and the projection of $\boldsymbol{b}$ to $\operatorname{Col}(b)$. We are given $\boldsymbol{b} = \begin{bmatrix} 4 & 0 & 6 & 2 \end{bmatrix}^\intercal$ and the projection of $\boldsymbol{b}$ to $\operatorname{Col}(B)$ is $\begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix}^\intercal$, so our error is

$$E = \left\| \begin{bmatrix} 4 \\ 0 \\ 6 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} 5 \\ 1 \\ 5 \\ 1 \end{bmatrix} \right\|^2 = 52$$

(12 pts) **Problem 4.** Find the value of $y_0$ so that the least squares line of best fit to the data set

$$\{(-3, y_0), (-2, -1), (-4, 1)\}$$

is $\widehat{f}(t) = -2 - t$. *Clearly explain your reasoning to receive credit.*

**Solution.** A "perfect" line $f(t) = a_0 + a_1 t$ would satisfy

$$\begin{array}{rcrcrcrcr} f(-3) & = & y_0 & \to & a_0 & - & 3\,a_1 & = & y_0 \\ f(-2) & = & -1 & \to & a_0 & - & 2\,a_1 & = & -1 \\ f(-4) & = & 1 & \to & a_0 & - & 4\,a_1 & = & 1 \end{array}$$

This is the system $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -4 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} y_0 \\ -1 \\ 1 \end{bmatrix}$$

The associated least squares problem is $A^\intercal A\widehat{x} = A^\intercal \boldsymbol{b}$. The relevant data here is

$$\overbrace{\begin{bmatrix} 1 & 1 & 1 \\ -3 & -2 & -4 \end{bmatrix}}^{A^\intercal}\overbrace{\begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -4 \end{bmatrix}}^{A} = \begin{bmatrix} 3 & -9 \\ -9 & 29 \end{bmatrix} \qquad \widehat{x} = \begin{bmatrix} \widehat{a}_0 \\ \widehat{a}_1 \end{bmatrix} \qquad \overbrace{\begin{bmatrix} 1 & 1 & 1 \\ -3 & -2 & -4 \end{bmatrix}}^{A^\intercal}\overbrace{\begin{bmatrix} y_0 \\ -1 \\ 1 \end{bmatrix}}^{\boldsymbol{b}} = \begin{bmatrix} y_0 \\ -3\,y_0 - 2 \end{bmatrix}$$

The key to this problem is that we are told that the least squares line of best fit is $\widehat{f}(t) = -t - 2 = \widehat{a}_0 + \widehat{a}_1\,t$, which means that $\widehat{a}_0 = -2$ and $\widehat{a}_1 = -1$. The quantity $A^\intercal A\widehat{x}$ is now

$$\overbrace{\begin{bmatrix} 3 & -9 \\ -9 & 29 \end{bmatrix}}^{A^\intercal A}\overbrace{\begin{bmatrix} -2 \\ -1 \end{bmatrix}}^{\widehat{\boldsymbol{x}}} = \begin{bmatrix} 3 \\ -11 \end{bmatrix}$$

Now, setting the vector $A^\intercal A\widehat{\boldsymbol{x}}$ equal to $A^\intercal \boldsymbol{b}$ gives the equations

$$3 = y_0 \qquad\qquad -11 = -3\,y_0 - 2$$

The value $y_0 = 3$ works in both equations. This is our $y_0$!