Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: ______________________ *Unique ID:* ______________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

October 25, 2024

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

Problem 1. Suppose that $A\boldsymbol{x} = \boldsymbol{b}$ is a *consistent system* where A is 31×17 and that $\boldsymbol{y}$ and $\boldsymbol{z}$ are solutions to this system where $\boldsymbol{y} \neq \boldsymbol{z}$ (this means that $\boldsymbol{y}$ and $\boldsymbol{z}$ are vectors satisfying $A\boldsymbol{y} = \boldsymbol{b}$ and $A\boldsymbol{z} = \boldsymbol{b}$).

(3 pts) (*a*) The vector $\boldsymbol{b}$ has _______ coordinates and the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ have _______ coordinates.

(4 pts) (*b*) Only one of the following statements is *guaranteed* to be correct. Select this statement.

○ $\boldsymbol{b}$ belongs to the row space of A ○ $\boldsymbol{b}$ belongs to the null space of A

○ $\boldsymbol{b}$ belongs to the column space of A ○ $\boldsymbol{b}$ belongs to the left null space of A

○ we do not have enough information to determine if $\boldsymbol{b}$ belongs to one of the four fundamental subspaces of A

(4 pts) (*c*) Only one of the following statements is *guaranteed* to accurately characterize $A^\intercal A\boldsymbol{y}$. Select this statement.

○ $A^\intercal A\boldsymbol{y}$ is a vector in the row space of A ○ $A^\intercal A\boldsymbol{y}$ is a vector in the null space of A

○ $A^\intercal A\boldsymbol{y}$ is a vector in the column space of A ○ $A^\intercal A\boldsymbol{y}$ is a vector in the left null space of A

(8 pts) (*d*) Prove that $\boldsymbol{y} - \boldsymbol{z} \in \operatorname{Null}(A^\intercal A)$. Use the space outside the box below to brainstorm your thoughts, but your proof must be neatly and succinctly presented in the box below. **You must avoid circular reasoning to receive credit.**

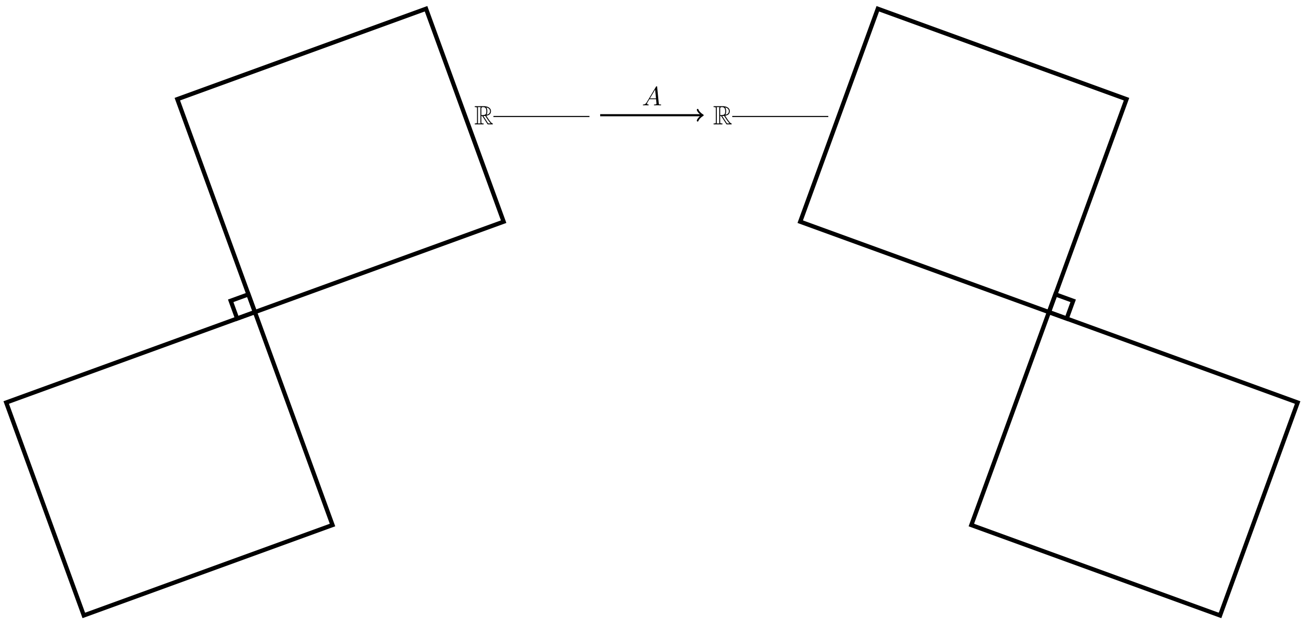
(9 pts) (*e*) The facts that $\boldsymbol{y} \neq \boldsymbol{z}$ and that $\boldsymbol{y} - \boldsymbol{z} \in \operatorname{Null}(A^\intercal A)$ imply that some, but not all, of the following statements are *guaranteed* to be true. Select the true statements (1.5pts each).

○ $\dim \operatorname{Null}(A^\intercal A) > 0$ ○ $\dim \operatorname{Null}(A^\intercal A) = 1$ ○ $A^\intercal A$ has independent columns

○ $A^\intercal A$ is singular ○ $\lambda = 0$ is *not* and eigenvalue of A ○ $\boldsymbol{y} - \boldsymbol{z}$ is an eigenvector of $A^\intercal A$

Problem 2. Suppose A satisfies $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\operatorname{rref}(A^\intercal) = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(10 pts) (a) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(3 pts) (b) Which of the following is the most accurate geometric description of the *left null space* of A?

○ a plane in $\mathbb{R}^6$ ○ a point with six coordinates ○ a line in $\mathbb{R}^7$ ○ a line in $\mathbb{R}^6$ ○ a plane in $\mathbb{R}^7$

(4 pts) (c) Only one of the following statements accurately describes the columns of A. Select this statement.

○ A has independent columns ○ every choice of five columns of A will be independent

○ it is impossible to find six independent vectors among the columns of A

○ the first four columns of A form a basis of $\operatorname{Col}(A)$

(4 pts) (d) The projection matrix P onto the row space of A is $P = X(X^\intercal X)^{-1}X^\intercal$ where X is ______ $\times$ ______.

(10 pts) (e) Let V be the vector space spanned by the rows of $\operatorname{rref}(A^\intercal)$. Find a basis of $V^\perp$. Clearly explain your work to receive credit.

Problem 3. One of the eigenvalues of $A = \begin{bmatrix} 5 & -5 & 0 & -5 \\ 25 & 0 & 25 & -5 \\ -5 & 5 & 0 & 5 \\ -25 & 0 & -25 & 5 \end{bmatrix}$ is $\lambda = 5$.

(10 pts) (*a*) Find a basis of $\mathcal{E}_A(5)$. Clearly explain your work to receive credit.

(9 pts) (*b*) Let P be the projection matrix onto $\mathcal{E}_A(5)$ and suppose that $\boldsymbol{v} \in \mathcal{E}_A(5)$. Some, but not all, of the following statements are correct. Select each correct statement (1.5pts each).

○ $A\boldsymbol{v} = 5 \cdot \boldsymbol{v}$ ○ $P\boldsymbol{v} = 5 \cdot \boldsymbol{v}$ ○ $A\boldsymbol{v} = \boldsymbol{O}$ ○ $P\boldsymbol{v} = \boldsymbol{v}$ ○ $\operatorname{trace}(P) = \operatorname{gm}_A(5)$ ○ $\operatorname{trace}(P) = 5$

(10 pts) (*c*) Let $\boldsymbol{b} = \begin{bmatrix} 4 & 0 & 6 & 2 \end{bmatrix}^\intercal$. If we assemble the basis vectors found for $\mathcal{E}_A(5)$ in part (*a*) of this problem into the columns of a matrix B, then we would find that $B(B^\intercal B)^{-1}B^\intercal \boldsymbol{b} = \begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix}^\intercal$. Use this information to calculate the error in the least squares problem associated to the system $B\boldsymbol{x} = \boldsymbol{b}$ and fill in the blank: $E =$ ________. Clearly explain your work to receive credit.

(12 pts) **Problem 4.** Find the value of y_0 so that the least squares line of best fit to the data set

$$\{(-3, y_0), (-2, -1), (-4, 1)\}$$

is $\widehat{f}(t) = -2 - t$. *Clearly explain your reasoning to receive credit.*