Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: Solutions

Unique ID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

November 22, 2024

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke University MATH

Problem 1. The data below depicts a matrix Q with orthonormal columns, the adjugate of an upper triangular matrix R, and the characteristic polynomial of R.

$$Q = \frac{1}{5}\begin{bmatrix} 2 & -2 & 3 \\ -2 & -4 & -2 \\ -1 & 0 & 2 \\ 4 & -1 & -2 \\ 0 & -2 & 2 \end{bmatrix} \qquad \operatorname{adj}(R) = \begin{bmatrix} -20 & -20 & 10 \\ 0 & 20 & -20 \\ 0 & 0 & -25 \end{bmatrix} \qquad \chi_R(t) = t^3 + 4t^2 - 25t - 100$$

Let $A = QR$. ***Do not ignore the factor of*** $1/5$ ***used to define*** Q.

(6 pts) (a) $\operatorname{trace}(Q^\intercal Q) =$ __3__ and $\operatorname{trace}(R) =$ __-4__

(4 pts) (b) The $(3, 1)$ cofactor of R is __10__.

(5 pts) (c) Each of the following statements is true. However, only one is equivalent to the statement "R^{-1} exists." Select this statement.

○ $\operatorname{trace}(\operatorname{adj}(R)) \neq 0$ ○ The eigenvalues of R do not sum to zero. ○ $\chi_R(t)$ is monic.

✓ $\chi_R(0) \neq 0$ ○ $\operatorname{adj}(R)$ exists.

(12 pts) (d) Let $\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 500 & 0 & 0 \end{bmatrix}^\intercal$. The system $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent. Find the least squares approximate solution $\widehat{\boldsymbol{x}}$ to $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your reasoning to receive credit.

Solution. The relevant feature of $A = QR$ is that the least squares problem associated to $A\boldsymbol{x} = \boldsymbol{b}$ reduces to $R\widehat{\boldsymbol{x}} = Q^\intercal \boldsymbol{b}$.

We are given $\operatorname{adj}(R)$ and we also know that $\det(R) = (-1)^3 \cdot \chi_R(0) = -(-100) = 100$. The adjugate formula for inverses then allows us to solve for $\widehat{\boldsymbol{x}}$ with

$$\widehat{\boldsymbol{x}} = \overbrace{\frac{1}{100}\begin{bmatrix} -20 & -20 & 10 \\ 0 & 20 & -20 \\ 0 & 0 & -25 \end{bmatrix}}^{R^{-1}} \overbrace{\frac{1}{5}\begin{bmatrix} 2 & -2 & -1 & 4 & 0 \\ -2 & -4 & 0 & -1 & -2 \\ 3 & -2 & 2 & -2 & 2 \end{bmatrix}}^{Q^\intercal} \overbrace{\begin{bmatrix} 0 \\ 0 \\ 500 \\ 0 \\ 0 \end{bmatrix}}^{\boldsymbol{b}}$$

$$= \begin{bmatrix} -20 & -20 & 10 \\ 0 & 20 & -20 \\ 0 & 0 & -25 \end{bmatrix} \begin{bmatrix} 2 & -2 & -1 & 4 & 0 \\ -2 & -4 & 0 & -1 & -2 \\ 3 & -2 & 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -20 & -20 & 10 \\ 0 & 20 & -20 \\ 0 & 0 & -25 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 40 \\ -40 \\ -50 \end{bmatrix}$$

(18 pts) **Problem 2.** Let $\boldsymbol{u}(t)$ be the solution to the initial value problem $\boldsymbol{u}' = A\boldsymbol{u}$ with $\boldsymbol{u}(0) = \boldsymbol{u}_0$ where

$$A = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} \qquad \boldsymbol{u}_0 = \begin{bmatrix} -5 \\ 15 \end{bmatrix}$$

Find $\boldsymbol{u}(t)$. Clearly explain your reasoning to receive credit.

Solution. From class we know that $\boldsymbol{u}(t) = \exp(At)\boldsymbol{u}_0$. Calculating matrix exponentials is hard, so we hope we can diagonalize $A = XDX^{-1}$, which gives $\boldsymbol{u}(t) = X\exp(Dt)X^{-1}$.

The characteristic polynomial of A is

$$\chi_A(t) = t^2 - \operatorname{trace}(A)\,t + \det(A) = t^2 + 7t + 6 = (t+1)\cdot(t+6)$$

This tells us that E-Vals$(A) = \{-1, -6\}$. The eigenspaces are

$$\mathcal{E}_A(-1) = \operatorname{Null}\overbrace{\begin{bmatrix} 4 & -1 \\ -4 & 1 \end{bmatrix}}^{-1\cdot I_2 - A} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\} \qquad \mathcal{E}_A(-6) = \operatorname{Null}\overbrace{\begin{bmatrix} -1 & -1 \\ -4 & -4 \end{bmatrix}}^{-6\cdot I_2 - A} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

Our diagonalization is then $A = XDX^{-1}$ where

$$X = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} \qquad X^{-1} = \frac{1}{-5}\begin{bmatrix} -1 & -1 \\ -4 & 1 \end{bmatrix}$$

Here, we have used the adjugate formula for inverses to calculate X^{-1}. Putting all this together gives

$$\begin{aligned}
\boldsymbol{u}(t) &= \overbrace{\begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}}^{X}\overbrace{\begin{bmatrix} e^{(-t)} & 0 \\ 0 & e^{(-6\,t)} \end{bmatrix}}^{\exp(Dt)}\overbrace{\frac{1}{-5}\begin{bmatrix} -1 & -1 \\ -4 & 1 \end{bmatrix}}^{X^{-1}}\overbrace{\begin{bmatrix} -5 \\ 15 \end{bmatrix}}^{\boldsymbol{u}_0} \\
&= \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} e^{(-t)} & 0 \\ 0 & e^{(-6\,t)} \end{bmatrix}\begin{bmatrix} 2 \\ -7 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 2\,e^{(-t)} \\ -7\,e^{(-6\,t)} \end{bmatrix} \\
&= \begin{bmatrix} 2\,e^{(-t)} - 7\,e^{(-6\,t)} \\ 8\,e^{(-t)} + 7\,e^{(-6\,t)} \end{bmatrix}
\end{aligned}$$

Problem 3. Suppose that $x > 0$ is a real number and consider the spectral factorization $H = UDU^*$ given by

$$\overbrace{\begin{bmatrix} ? & -i & 1-i & i \\ i & ? & -1 & 1-i \\ 1+i & -1 & ? & -1 \\ -i & 1+i & -1 & ? \end{bmatrix}}^{H} = \overbrace{\begin{bmatrix} \frac{1}{\sqrt{x}} & \frac{1}{2} & \frac{1}{2} & \frac{5}{\sqrt{60}} \\ \frac{-2}{\sqrt{x}} & \frac{1+i}{2} & 0 & \frac{-1-3i}{\sqrt{60}} \\ \frac{2+i}{\sqrt{x}} & \frac{i}{2} & \frac{-i}{2} & \frac{-2-i}{\sqrt{60}} \\ \frac{-1-i}{\sqrt{x}} & 0 & \frac{1-i}{2} & \frac{-2+4i}{\sqrt{60}} \end{bmatrix}}^{U} \overbrace{\begin{bmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{bmatrix}}^{D} \begin{bmatrix} \\ & & U^* & \\ \\ \end{bmatrix}$$

Note that each entry in the first column of U has a denominator of $\sqrt{x}$ and that every diagonal entry of H is unknown and marked ?.

(5 pts) (*a*) $x =$ <u>12</u>

Solution. Every column of U must be a unit vector, so we need

$$1 = \left\| (1/\sqrt{x}) \cdot \begin{bmatrix} 1 & -2 & 2+i & -1-i \end{bmatrix}^{\intercal} \right\|^2 = (1/x) \cdot (1^2 + (-2)^2 + 2^2 + 1^2 + (-1)^2 + (-1)^2) = 12/x$$

Hence $x = 12$.

(16 pts) (*b*) Which of the following scalars is equal to zero? Select all that apply (2pts each).

√ $\operatorname{trace}(H)$ ○ $\det(H)$ √ $\operatorname{nullity}(i \cdot I_4 - H)$ √ The "imaginary part" of every entry marked ?.

√ $\chi_H(i^2)$ √ $\det(3 \cdot I_4 - H)$ ○ One of the eigenvalues of U.

√ The "real part" of $\exp(h_{31} \cdot \pi/2) = e^{h_{31} \cdot \pi/2}$ where h_{31} is the $(3, 1)$ entry of H.

Solution. First, $\operatorname{trace}(H) = \operatorname{trace}(D) = 3 + 1 - 1 - 3 = 0$. Then, $\det(H) = \det(D) = 3 \cdot (1) \cdot (-1) \cdot (-3) \neq 0$. Since $\lambda = i$ is not an eigenvalue of H, the characteristic matrix $i \cdot I_4 - H$ is *nonsingular*, which means $\operatorname{nullity}(i \cdot I_4 - H) = 0$. Next, since $H^* = H$, every diagonal entry ? of H must satisfy $\overline{?} = ?$, which means every ? is a real number. Since $i^2 = -1$ is an eigenvalue of H, $\chi_H(i^2) = 0$. Similarly, $\det(3 \cdot I_4 - H) = 0$. We know $U^* = U^{-1}$, so none of the eigenvalues of U is zero. Finally, Euler's formula tells us that

$$\exp(h_{31} \cdot \pi/2) = \exp((\pi/2) + (\pi/2) \cdot i) = e^{\pi/2}\{\cos(\pi/2) + i \sin(\pi/2)\} = e^{\pi/2}\{0 + i \cdot 1\} = i\, e^{\pi/2}$$

So, the "real part" of this number is zero.

(10 pts) (*c*) Suppose that Y is invertible and let $A = (U^* Y^{-1}) D (YU)$. Calculate $\chi_A(2)$. Clearly explain your reasoning to receive credit.

Solution. Let $X = U^* Y^{-1}$ and note that $X(YU) = U^* Y^{-1} Y U = U^* I U = U^* U = I$. This means $X^{-1} = YU$ so $A = XDX^{-1}$. It follows that

$$\chi_A(t) = (t-3)(t-1)(t+1)(t+3)$$

Hence

$$\chi_A(2) = (2-3)(2-1)(2+1)(2+3) = (-1)(1)(3)(5) = -15$$

Problem 4. Let S be the 4×4 real symmetric matrix that defines the quadratic form

$$q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x}\rangle = 5\,x_1^2 - 2\,x_1x_2 + x_2^2 + 8\,x_1x_3 + 4\,x_2x_3 + 4\,x_3^2 + 6\,x_1x_4 - 2\,x_2x_4 + 8\,x_3x_4 + 5\,x_4^2$$

If we "complete the square", then this quadratic form simplifies to

$$q(\boldsymbol{x}) = \lambda_1\,y_1^2 + 3\,y_2^2 + 2\,y_3^2 - 2\,y_4^2$$

where λ_1 is a scalar and y_1, y_2, y_3 are expressions that depend on x_1, x_2, x_3.

(4 pts) (*a*) What is the definiteness of S? Select all that apply (no partial credit here).

○ positive definite ○ positive semidefinite ○ negative definite ○ negative semidefinite √ indefinite

(5 pts) (*b*) Only one of the following expressions is a correct formula for the quadratic form $f(\boldsymbol{x}) = \langle \boldsymbol{x}, \exp(S)\boldsymbol{x}\rangle$. Select this expression.

√ $f(\boldsymbol{x}) = e^{\lambda_1}\,y_1^2 + e^3\,y_2^2 + e^2\,y_3^2 + e^{-2}\,y_4^2$ ○ $f(\boldsymbol{x}) = e^{\lambda_1\,y_1^2}e^{3\,y_2^2}e^{2\,y_3^2}e^{-2\,y_4^2}$

○ $f(\boldsymbol{x}) = e^{\lambda_1\,y_1^2} + e^{3\,y_2^2} + e^{2\,y_3^2} + e^{-2\,y_4^2}$ ○ $f(\boldsymbol{x}) = \lambda_1\,e^{y_1^2} + 3\,e^{y_2^2} + 2\,e^{y_3^2} - 2\,e^{y_4^2}$ ○ none of these

(5 pts) (*c*) $\lambda_1 =$ __12__

Solution. The formula $q(\boldsymbol{x}) = \lambda_1\,y_1^2 + 3\,y_2^2 + 2\,y_3^2 - 2\,y_4^2$ from "completing the square" tells us that E-Vals$(S) = \{\lambda_1, 3, 2, -2\}$. The "square" terms in the first formula for $q(\boldsymbol{x})$ are $5\,x_1^2, x_2^2, 4\,x_3^2, 5\,x_4^2$, which means that the diagonal entries of S are $5, 1, 4, 5$. The eigenvalue formula for trace then tells us that

$$\lambda_1 + 3 + 2 - 2 = \operatorname{trace}(S) = 5 + 1 + 4 + 5 = 15$$

It follows that $\lambda_1 = 15 - 3 - 2 + 2 = 12$.

(10 pts) (*d*) Calculate y_3^2 when $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, and $x_4 = 8$. Clearly explain your reasoning to receive credit.

Solution. The technique of "completing the square" expresses our quadratic form as

$$q(\boldsymbol{x}) = \lambda_1\,y_1^2 + 3\,y_2^2 + 2\,y_3^2 - 2\,y_4^2$$

Here, E-Vals$(S) = \{\lambda_1, 3, 2, -2\}$ and $\boldsymbol{y} = U^\intercal\boldsymbol{x}$ where $S = UDU^\intercal$ is a spectral factorization. This spectral factorization looks like

$$\overset{S}{\begin{bmatrix} 5 & -1 & 4 & 3 \\ -1 & 1 & 2 & -1 \\ 4 & 2 & 4 & 4 \\ 3 & -1 & 4 & 5 \end{bmatrix}} = \overset{U}{\begin{bmatrix} | & | & | & | \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \boldsymbol{u}_4 \\ | & | & | & | \end{bmatrix}} \overset{D}{\begin{bmatrix} \lambda_1 & & & \\ & 3 & & \\ & & 2 & \\ & & & -2 \end{bmatrix}} \overset{U^\intercal}{\begin{bmatrix} \text{---} & \boldsymbol{u}_1^\intercal & \text{---} \\ \text{---} & \boldsymbol{u}_2^\intercal & \text{---} \\ \text{---} & \boldsymbol{u}_3^\intercal & \text{---} \\ \text{---} & \boldsymbol{u}_4^\intercal & \text{---} \end{bmatrix}}$$

We know that $y_3 = \langle \boldsymbol{u}_3, \boldsymbol{x}\rangle$ and that $\boldsymbol{u}_3$ is a unit basis vector of $\mathcal{E}_S(2)$. We then turn our attention to the eigenspace

$$\mathcal{E}_S(2) = \operatorname{Null}\overset{2\cdot I_4 - S}{\begin{bmatrix} -3 & 1 & -4 & -3 \\ 1 & 1 & -2 & 1 \\ -4 & -2 & -2 & -4 \\ -3 & 1 & -4 & -3 \end{bmatrix}} = \operatorname{Span}\left\{\boldsymbol{u}_3 = \frac{\pm 1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\right\}$$

Here, the two valid formulas for $\boldsymbol{u}_3$ were found by noticing the column relation $\text{Col}_1 = \text{Col}_4$ in the characteristic matrix $2\cdot I_4 - S$. Now, we have

$$y_3^2 = \left(\frac{\pm(x_1 - x_4)}{\sqrt{2}}\right)^2 = \left(\frac{\pm(2-8)}{\sqrt{2}}\right)^2 = \left(\frac{\pm 6}{\sqrt{2}}\right)^2 = \frac{36}{2} = 18$$