Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: ______________________ *Unique ID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

November 22, 2024

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. The data below depicts a matrix Q with orthonormal columns, the adjugate of an upper triangular matrix R, and the characteristic polynomial of R.

$$Q = \frac{1}{5}\begin{bmatrix} 2 & -2 & 3 \\ -2 & -4 & -2 \\ -1 & 0 & 2 \\ 4 & -1 & -2 \\ 0 & -2 & 2 \end{bmatrix} \qquad \operatorname{adj}(R) = \begin{bmatrix} -20 & -20 & 10 \\ 0 & 20 & -20 \\ 0 & 0 & -25 \end{bmatrix} \qquad \chi_R(t) = t^3 + 4t^2 - 25t - 100$$

Let $A = QR$. ***Do not ignore the factor of*** $^1/_5$ ***used to define*** Q.

(6 pts) (a) $\operatorname{trace}(Q^\intercal Q) =$ _______ and $\operatorname{trace}(R) =$ _______

(4 pts) (b) The $(3,1)$ cofactor of R is _______.

(5 pts) (c) Each of the following statements is true. However, only one is equivalent to the statement "R^{-1} exists." Select this statement.

○ $\operatorname{trace}(\operatorname{adj}(R)) \neq 0$ ○ The eigenvalues of R do not sum to zero. ○ $\chi_R(t)$ is monic.

○ $\chi_R(0) \neq 0$ ○ $\operatorname{adj}(R)$ exists.

(12 pts) (d) Let $\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 500 & 0 & 0 \end{bmatrix}^\intercal$. The system $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent. Find the least squares approximate solution $\widehat{\boldsymbol{x}}$ to $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your reasoning to receive credit.

(18 pts) **Problem 2.** Let $\boldsymbol{u}(t)$ be the solution to the initial value problem $\boldsymbol{u}' = A\boldsymbol{u}$ with $\boldsymbol{u}(0) = \boldsymbol{u}_0$ where

$$A = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} \qquad \boldsymbol{u}_0 = \begin{bmatrix} -5 \\ 15 \end{bmatrix}$$

Find $\boldsymbol{u}(t)$. Clearly explain your reasoning to receive credit.

Problem 3. Suppose that $x > 0$ is a real number and consider the spectral factorization $H = UDU^*$ given by

$$\overset{H}{\begin{bmatrix} ? & -i & 1-i & i \\ i & ? & -1 & 1-i \\ 1+i & -1 & ? & -1 \\ -i & 1+i & -1 & ? \end{bmatrix}} = \overset{U}{\begin{bmatrix} \frac{1}{\sqrt{x}} & \frac{1}{2} & \frac{1}{2} & \frac{5}{\sqrt{60}} \\ \frac{-2}{\sqrt{x}} & \frac{1+i}{2} & 0 & \frac{-1-3i}{\sqrt{60}} \\ \frac{2+i}{\sqrt{x}} & \frac{i}{2} & \frac{-i}{2} & \frac{-2-i}{\sqrt{60}} \\ \frac{-1-i}{\sqrt{x}} & 0 & \frac{1-i}{2} & \frac{-2+4i}{\sqrt{60}} \end{bmatrix}} \overset{D}{\begin{bmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{bmatrix}} \begin{bmatrix} & & & \\ & U^* & & \\ & & & \end{bmatrix}$$

Note that each entry in the first column of U has a denominator of $\sqrt{x}$ and that every diagonal entry of H is unknown and marked ?.

(5 pts) (*a*) $x =$ _______

(16 pts) (*b*) Which of the following scalars is equal to zero? Select all that apply (2pts each).

○ trace(H) ○ det(H) ○ nullity$(i \cdot I_4 - H)$ ○ The "imaginary part" of every entry marked ?.

○ $\chi_H(i^2)$ ○ det$(3 \cdot I_4 - H)$ ○ One of the eigenvalues of U.

○ The "real part" of $\exp(h_{31} \cdot \pi/2) = e^{h_{31} \cdot \pi/2}$ where h_{31} is the $(3,1)$ entry of H.

(10 pts) (*c*) Suppose that Y is invertible and let $A = (U^*Y^{-1})D(YU)$. Calculate $\chi_A(2)$. Clearly explain your reasoning to receive credit.

Problem 4. Let S be the 4×4 real symmetric matrix that defines the quadratic form

$$q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle = 5\,x_1^2 - 2\,x_1x_2 + x_2^2 + 8\,x_1x_3 + 4\,x_2x_3 + 4\,x_3^2 + 6\,x_1x_4 - 2\,x_2x_4 + 8\,x_3x_4 + 5\,x_4^2$$

If we "complete the square", then this quadratic form simplifies to

$$q(\boldsymbol{x}) = \lambda_1\,y_1^2 + 3\,y_2^2 + 2\,y_3^2 - 2\,y_4^2$$

where λ_1 is a scalar and y_1, y_2, y_3 are expressions that depend on x_1, x_2, x_3.

(4 pts) (*a*) What is the definiteness of S? Select all that apply (no partial credit here).

○ positive definite ○ positive semidefinite ○ negative definite ○ negative semidefinite ○ indefinite

(5 pts) (*b*) Only one of the following expressions is a correct formula for the quadratic form $f(\boldsymbol{x}) = \langle \boldsymbol{x}, \exp(S)\boldsymbol{x} \rangle$. Select this expression.

○ $f(\boldsymbol{x}) = e^{\lambda_1}\,y_1^2 + e^3\,y_2^2 + e^2\,y_3^2 + e^{-2}\,y_4^2$ ○ $f(\boldsymbol{x}) = e^{\lambda_1\,y_1^2}e^{3\,y_2^2}e^{2\,y_3^2}e^{-2\,y_4^2}$

○ $f(\boldsymbol{x}) = e^{\lambda_1\,y_1^2} + e^{3\,y_2^2} + e^{2\,y_3^2} + e^{-2\,y_4^2}$ ○ $f(\boldsymbol{x}) = \lambda_1\,e^{y_1^2} + 3\,e^{y_2^2} + 2\,e^{y_3^2} - 2\,e^{y_4^2}$ ○ none of these

(5 pts) (*c*) $\lambda_1 =$ ________

(10 pts) (*d*) Calculate y_3^2 when $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, and $x_4 = 8$. Clearly explain your reasoning to receive credit.