Matrices and Vectors

Name: ______________________________ NetID: _______________

Problem 1. Suppose that $x, y, z \in \mathbb{R}$ are fixed scalars and consider the equation

$$c_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

What are the only valid choices for c_1, c_2, c_3 that make this equation valid? **You are always expected to clearly (and concisely) explain your reasoning.**

Problem 2. Suppose that $x, y, z \in \mathbb{R}$ are fixed scalars and consider the equation

$$c_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Under what condition on x, y, and z is this equation possible and what are the only valid choices for c_1 c_2 if this condition is met?

Understanding how to efficiently demonstrate that equations hold will be reoccuring and important theme in this course. The next three problems are designed to help train you to efficiently work with matrix equations. Make sure to take the time to properly understand how to do these problems!

Problem 3. Suppose that $S = A + A^\intercal$ where A is 2023×2023. Show that S is *symmetric*. This can be done very quickly by filling in the blanks below.

$$S^\intercal = (A + A^\intercal)^\intercal = \underline{\hspace{4cm}} = \underline{\hspace{4cm}} = \underline{\hspace{4cm}}$$

Problem 4. Suppose that $S = S_1 + S_2$ where S_1 and S_2 are 2023×2023 symmetric matrices. Show that S is also symmetric. Again, this can be done very quickly by filling in the blanks below.

$$S^\intercal = (S_1 + S_2)^\intercal = \underline{\hspace{4cm}} = \underline{\hspace{4cm}} = \underline{\hspace{4cm}}$$

Problem 5. We call a matrix X *traceless* if $\operatorname{trace}(X) = 0$. Suppose that $A = c_1 \cdot A_1 + c_2 \cdot A_2$ where A_1 and A_2 are traceless.

(a) The set-up of this problem is telling us that $\operatorname{trace}(A_1) = \underline{\hspace{1.5cm}}$ and $\operatorname{trace}(A_2) = \underline{\hspace{1.5cm}}$.

(b) Trace is a *linear operation*, which means that $\operatorname{trace}(c_1 \cdot A_1 + c_2 \cdot A_2) = \underline{\hspace{5cm}}$.

(c) Show that A is traceless. Like in the preceeding two problems, this can (and absolutely should) be done quickly with a single equation.

Problem 6. It is known that the columns of the matrix A below satisfy the equation on the right.

$$A = \begin{bmatrix} 1 & -2 & 2 & * \\ 0 & 1 & 0 & * \\ 0 & 2 & 1 & * \end{bmatrix} \qquad\qquad 2 \cdot \mathrm{Col}_1 - \mathrm{Col}_2 + 3 \cdot \mathrm{Col}_3 - \mathrm{Col}_4 = \boldsymbol{O}$$

Use this information to find the missing column of A.

Matrix-Vector Products

Name: ______________________________ NetID: _______________

Problem 1. Consider the following linear combination of vectors

$$-13 \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 5 \cdot \begin{bmatrix} 0 \\ 9 \\ 3 \end{bmatrix} + 15 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (-11) \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + 9 \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Write this linear combination as a matrix-vector product $A\boldsymbol{v}$. What is the size of A?

Problem 2. Suppose A is 2022×3. What vector $\boldsymbol{v}$ would we multiply A by if we wanted to subtract four times the third column of A from the first column of A? Explain.

Problem 3. Suppose A is 3×2022. What matrix-vector product subtracts the third row of A from the first row of A? *Hint.* The rows of A are the columns of what matrix?

Problem 4. Suppose that A is 4×7 and that $\boldsymbol{v} \in \mathbb{R}^7$. Define $\boldsymbol{w} = A\boldsymbol{v}$. If B is a matrix such that $B\boldsymbol{w} \in \mathbb{R}^5$, then what is the size of B?

Problem 5. Suppose that A is $n \times n$ and consider $\boldsymbol{v} = c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2$ where $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{E}_A(\lambda)$. Show that $\boldsymbol{v} \in \mathcal{E}_A(\lambda)$. This can be quickly done by filling in the blanks below.

$$
\begin{aligned}
A\boldsymbol{v} &= A(c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2) \\
&= \underline{\hspace{8cm}} \\
&= \underline{\hspace{8cm}} \\
&= \underline{\hspace{8cm}} \\
&= \underline{\hspace{3cm}}
\end{aligned}
$$

Problem 6. Suppose that A is $n \times n$ and that $\boldsymbol{v} \in \mathcal{E}_A(\lambda)$. Calculate $X_\lambda \boldsymbol{v}$ where $X_\lambda = \lambda \cdot I_n - A$.

Problem 7. Consider the matrix A and the vector $\boldsymbol{v}$ given by

$$A = \begin{bmatrix} 1 & 2 & -5 & -3 \\ -4 & -8 & 20 & * \\ -1 & -2 & 5 & * \\ 0 & 0 & 0 & * \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 1 \\ v_2 \\ -1 \\ 0 \end{bmatrix}$$

Note that both A and $\boldsymbol{v}$ are missing entries!

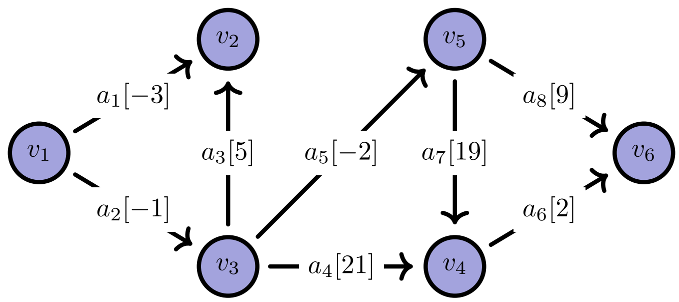
(a) If we assume that A has rank one, then what are the missing entries of A?

(b) If we assume that $\boldsymbol{v}$ is an eigenvector of A, then there are exactly two possibilities for the missing entry v_2 of $\boldsymbol{v}$. Find the two possible values of v_2. In each case, what is the corresponding eigenvalue?

Digraphs

Name: ______________________________ NetID: ________________

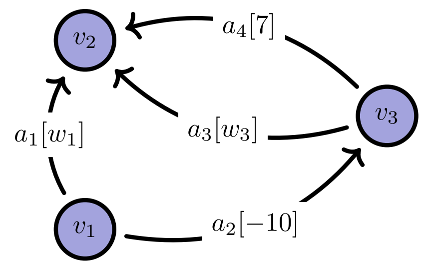
Problem 1. Let G be the weighted digraph given by



(a) Use a matrix-vector product to calculate the weighted net flow through all of the nodes of G. Do the calculation on a computer, but show the matrix and the vector being multiplied.

(b) Let $\boldsymbol{v}$ be the cycle vector corresponding to the cycle (a_6, a_8, a_5, a_4). Calculate $A\boldsymbol{v}$ where A is the incidence matrix of G. Do the calculation on a computer, but show the matrix and the vector being multiplied. *Note.* The weight vector in this part of the problem is completely different than the weight vector in part (a). Here, $\boldsymbol{v}$ is a *cycle vector*, so every entry should either equal 0, 1, or -1.

Problem 2. Let G be the weighted digraph given by



Note that the two weights w_1 and w_3 are unknown.

(a) Suppose that the weighted net flows through the nodes are given by the vector $\boldsymbol{b} = \begin{bmatrix} 18 & -10 & -8 \end{bmatrix}^\intercal$. Find the missing weights w_1 and w_3.

(b) Is it possible that the weighted net flows through the nodes could instead be given by $\boldsymbol{b} = \begin{bmatrix} 2 & 2 & -1 \end{bmatrix}^\intercal$? Explain why or why not.

Vector Geometry

Name: ______________________________ NetID: ________________

Problem 1. Suppose $\|\boldsymbol{v}\| = 4$ and $\|\boldsymbol{w}\| = 7$. Use inner products to determine the largest and smallest possible values of $\|6\,\boldsymbol{v} - 5\boldsymbol{w}\|$. *Hint.* Start by expanding $\|6\,\boldsymbol{v} - 5\,\boldsymbol{w}\|^2$.

Problem 2. Suppose that $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are orthogonal to a vector $\boldsymbol{v}$. Show that every linear combination $c_1 \cdot \boldsymbol{w}_1 + c_2 \cdot \boldsymbol{w}_2$ is also orthogonal to $\boldsymbol{v}$.

Problem 3. Suppose that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are two vectors in $\mathbb{R}^n$ where $\boldsymbol{v}_1 \neq \mathbf{0}$. Let

$$\boldsymbol{w} = \boldsymbol{v}_2 - \frac{\langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle}{\langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle} \cdot \boldsymbol{v}_1$$

Show that $\boldsymbol{w}$ is orthogonal to $\boldsymbol{v}_1$.

Problem 4. Consider the vectors $\boldsymbol{v}$ nad $\boldsymbol{w}$ given by

$$\boldsymbol{v} = \begin{bmatrix} -3 & 1 & 0 \end{bmatrix}^\intercal \qquad \boldsymbol{w} = \begin{bmatrix} -1 & -2 & -23 \end{bmatrix}^\intercal$$

Suppose that A is a 3×3 symmetric matrix satisfying $A\boldsymbol{v} = \begin{bmatrix} -4 & 10 & 8 \end{bmatrix}^\intercal$. Find $\langle \boldsymbol{v}, A\boldsymbol{w} \rangle$.

Matrix Multiplication

Name: ________________________________ NetID: ________________

Note. You should always feel free to outsource matrix multiplication calculations to a computer. However, you must always make it clear which matrices you are multiplying in your work.

Problem 1. Consider the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ given by

$$\boldsymbol{v} = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 5 \\ 3 \\ -1 \\ -9 \end{bmatrix}$$

Here, we will think of $\boldsymbol{v}$ as a 3×1 matrix and $\boldsymbol{w}$ as a 4×1 matrix. The *outer product* of $\boldsymbol{v}$ and $\boldsymbol{w}$ is defined by the formula $\boldsymbol{v} \otimes \boldsymbol{w} = \boldsymbol{v}\boldsymbol{w}^{\intercal}$.

(a) Calculate $\boldsymbol{v} \otimes \boldsymbol{w}$ (note that this can be quickly done without a computer). What type of object is $\boldsymbol{v} \otimes \boldsymbol{w}$?

(b) Are $\boldsymbol{v} \otimes \boldsymbol{w}$ and $\boldsymbol{w} \otimes \boldsymbol{v}$ equal? If not, are they related?

Problem 2. Consider the matrices U and V given by

$$U = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \\ -2 & -2 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ be the columns of U and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be the columns of V.

(a) Calculate $A = \boldsymbol{u}_1 \otimes \boldsymbol{v}_1 + \boldsymbol{u}_2 \otimes \boldsymbol{v}_2 + \boldsymbol{u}_3 \otimes \boldsymbol{v}_3$.

(b) Calculate $B = UV^\intercal$. How does the matrix A from part (a) compare to B?

Problem 3. Suppose that A and B are $n \times n$ symmetric matrices and define $S = ABA$. Show that S is also symmetric.

Problem 4. Suppose that A is 2023×3, so A looks like

$$A = \begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix}$$

Suppose that the columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are all orthogonal to each other and that $\|\boldsymbol{a}_1\| = 1$, $\|\boldsymbol{a}_2\| = 2$, and $\|\boldsymbol{a}_3\| = 3$. Find the Gramian $A^\intercal A$ of A.

Problem 5. Consider the matrices A, B, and Y given by

$$A = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & -1 \\ -5 & -4 & 0 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} -2 & -2 & 0 \\ 2 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

(a) Calculate AB and BA. Are either of these matrices familiar?

(b) Find a matrix X satisfying $AX = Y$.

Row Echelon Forms

Name: ____ NetID: ____________________

Problem 1. Consider the matrix A, which is in reduced row echelon form and the vector $\boldsymbol{v}$ given by

$$A = \begin{bmatrix} 1 & 0 & 0 & -5 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} * \\ * \\ * \\ 1 \\ * \end{bmatrix}$$

Find the missing entries in $\boldsymbol{v}$ so that $A\boldsymbol{v} = \boldsymbol{0}$.

Hint. Note that the columns of A satisfy $\boldsymbol{a}_4 = -5 \cdot \boldsymbol{a}_1 - 3 \cdot \boldsymbol{a}_2 + 7 \cdot \boldsymbol{a}_3$.

Problem 2. Consider the matrix A, which is in reduced row echelon form and the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ given by

$$A = \begin{bmatrix} 1 & 0 & 7 & 0 & -9 \\ 0 & 1 & 4 & 0 & -2 \\ 0 & 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{v}_1 = \begin{bmatrix} * \\ * \\ 1 \\ * \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} * \\ * \\ 0 \\ * \\ 1 \end{bmatrix}$$

Find the missing entries in $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ so that $A\boldsymbol{v}_1 = \boldsymbol{0}$ and $A\boldsymbol{v}_2 = \boldsymbol{0}$.

Hint. The pivot columns of A are $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_4\}$ and the nonpivot columns of A are $\{\boldsymbol{a}_3, \boldsymbol{a}_5\}$. Start by expressing each nonpivot column as a linear combination of the pivot columns.

Problem 3. There is only one 4×6 matrix R in reduced row echelon form with column relations

$$\mathrm{Col}_2 = -2 \cdot \mathrm{Col}_1 \qquad \mathrm{Col}_5 = 2 \cdot \mathrm{Col}_1 + \mathrm{Col}_4 \qquad \mathrm{Col}_6 = \mathrm{Col}_1 - \mathrm{Col}_3 + 2 \cdot \mathrm{Col}_4$$

(a) What is $\mathrm{rank}(R)$?

(b) Find R.

Problem 4. Consider the matrix A, which is in row echelon form and the vector $\boldsymbol{v}$ given by

$$A = \begin{bmatrix} -1 & 1 & 1 & -1 & -1 \\ 0 & 6 & 2 & -4 & -3 \\ 0 & 0 & -4 & -2 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ -6 \\ 0 \end{bmatrix}$$

Find the missing entries in $\boldsymbol{v}$ so that $A\boldsymbol{v} = \boldsymbol{0}$.

Problem 5. Suppose A is 8×13 and in row echelon form. Is it possible for $\mathrm{rank}(A) = \mathrm{nullity}(A)$? Why or why not?

Linear Systems

Name: ______________________________ NetID: ________________

Problem 1. Consider the reduced row echelon form system given by

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & -2 & 4 & 7 & 9 \end{array}\right]$$

Find the general solution to this system.

Problem 2. Consider the reduced row echelon form system $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & 0 & -4 & 0 & -12 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ -8 \\ 4 \\ 0 \end{bmatrix}$$

Find the general solution to this system.

Problem 3. Consider the reduced row echelon form system $R\boldsymbol{x} = \boldsymbol{b}$ where

$$R = \begin{bmatrix} 1 & -3 & 0 & 2 & -6 \\ 0 & 0 & 1 & 7 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -42 \\ 128 \\ 0 \\ 0 \end{bmatrix}$$

Write the general solution $\boldsymbol{x}$ to $R\boldsymbol{x} = \boldsymbol{b}$ as $\boldsymbol{x}_p + c_1 \cdot \boldsymbol{x}_1 + c_2 \cdot \boldsymbol{x}_2 + c_3 \cdot \boldsymbol{x}_3$. Calculate the matrix-vector products $R\boldsymbol{x}_1$, $R\boldsymbol{x}_2$, and $R\boldsymbol{x}_3$.

Problem 4. Consider the row echelon form system $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 2 & -4 & 5 & 3 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 6 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 4 \\ -2 \\ -12 \end{bmatrix}$$

Use back-substitution to find the general solution to this system.

Gauss-Jordan Elimination

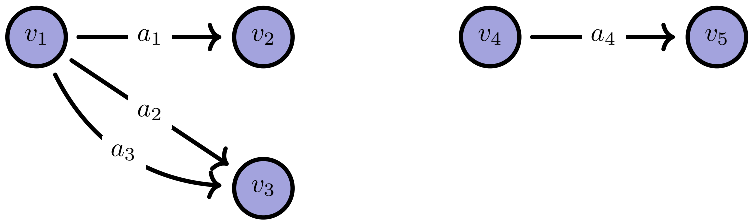
Name: ______________________________ NetID: ______________

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 3 & -9 & 0 & 0 & 15 \\ -2 & 6 & 0 & 0 & -10 \\ 7 & -21 & 4 & 0 & 79 \\ 9 & -27 & 2 & 8 & 51 \end{bmatrix}$$

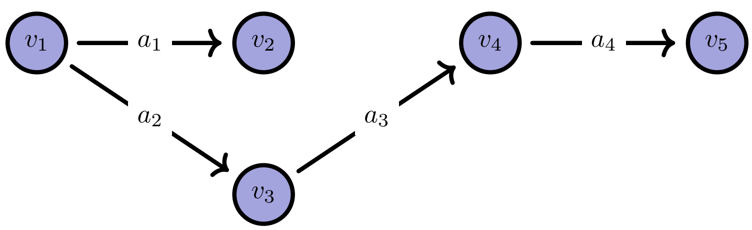
Calculate $\operatorname{rref}(A)$ by applying the Gauß-Jordan algorithm. You can (and should!) check your work on a computer, but you must precisely follow the algorithm (including labels on your row-reductions) and show each step here.

Problem 2. Let A be the incidence matrix of the digraph



Find $\operatorname{rref}(A)$ (on a computer, if you're confident you could do it by hand) and use this to calculate $\operatorname{nullity}(A)$ and $\operatorname{nullity}(A^\intercal)$.

Problem 3. Let A be the incidence matrix of the digraph



Find $\operatorname{rref}(A)$ (on a computer, if you're confident you could do it by hand) and use this to calculate $\operatorname{nullity}(A)$ and $\operatorname{nullity}(A^\intercal)$.

Problem 4. The matrix $A = \begin{bmatrix} 3 & 15 & -24 \\ -5 & -25 & 40 \\ 2 & 10 & -16 \\ -7 & -35 & 56 \end{bmatrix}$ is rank one. Find $\operatorname{rref}(A)$ *without doing any row reductions.*

Hint. Start by identifying the column relations.

Problem 5. Suppose that a matrix A satisfies

$$\operatorname{rref} \overset{A}{\begin{bmatrix} -9 & -43 & * & -133 & * \\ 3 & 16 & * & 48 & * \\ 5 & 25 & * & 76 & * \\ -5 & -24 & * & -74 & * \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 5 & 0 & 1 \\ 0 & 1 & -3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that two of the columns in A are currently unknown.

(a) Find the third column of A *without doing any row reductions.*

(b) Find the fifth column of A *without doing any row reductions.*

Nonsingular Matrices

Name: ________________________________ NetID: ____________________

Problem 1. Consider the matrix A, its inverse A^{-1}, and the vector $\boldsymbol{b}$ given by

$$A = \begin{bmatrix} -6 & -6 & 10 & 3 \\ 2 & 3 & -5 & -3 \\ 1 & 1 & -2 & -1 \\ -2 & -2 & 3 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 0 & 1 & -4 & -1 \\ 1 & 0 & 0 & -3 \\ -1 & 0 & -2 & 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} t \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

Find all solutions $\boldsymbol{x}$ to the system $A\boldsymbol{x} = \boldsymbol{b}$.

Problem 2. Consider the matrix L given by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

Find L^{-1}.

Problem 3. Suppose A is an $n \times n$ matrix that satisfies the equation

$$A^3 - 4\,A^2 + 7\,A - I_n = \boldsymbol{O}_n$$

Show that A is invertible and find a formula for A^{-1}.

Problem 4. Suppose A is a matrix satisfying

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} 1 \\ -7 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Is this enough information to determine if A is nonsingular? Why or why not?

In general, inverses are difficult to calculate because we do not have a convenient *formula* for them. This next problem investigates this matter for generic 2×2 matrices. If you successfully work through this problem, then you will be rewarded with a faster way of calculating 2×2 inverses!

Problem 5. The formulas below depict a generic 2×2 matrix A and another matrix B whose entries are related to the entries in A.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad\qquad B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The matrix B is called the *adjugate* of A.

(a) There is a scalar x satisfying the equation $AB = x \cdot I_2$. Find this value of x.

(b) If we know that the value of x from the previous part of this problem is nonzero, then what is the significance of the matrix $\frac{1}{x} \cdot B$? Use mathematical reasoning to explain.

Problem 6. Supose A satisfies the equation

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is this enough information to determine if $(A^\intercal A)^{-1}$ exists? Why or why not?

$EA = R$ Factorizations

Name: ______________________________ NetID: ________________

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 & 7 \\ -4 & -12 & 0 & 2 & -20 \\ 2 & 6 & 1 & -2 & -3 \end{bmatrix}$$

Use the Gauß-Jordan algorithm to find $EA = R$.

Problem 2. Consider the $EA = R$ factorization

$$\overset{E}{\begin{bmatrix} 20 & 2 & 45 & 0 \\ 52 & -14 & 109 & -5 \\ 2 & 1 & 5 & 0 \\ -13 & 2 & -28 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 11 & -44 & 44 & 35 & 57 & -59 \\ 3 & -12 & 12 & 10 & 16 & -17 \\ -5 & 20 & -20 & -16 & -26 & 27 \\ -3 & 12 & -12 & -13 & -19 & 23 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & -4 & 4 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

(a) Suppose that $\boldsymbol{b}$ is a vector that makes the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent. What system of equations must the coordinates of $\boldsymbol{b}$ solve and what is the coefficient matrix of this system?

(b) Let R' be the matrix obtained by deleting the last two rows of R. Find a matrix C satisfying $A = CR'$ without doing any row reductions.

$PA = LU$ Factorizations

Name: ____ NetID: ____________________

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 2 & -3 & 5 & 0 & 7 \\ -4 & 6 & -10 & 4 & -7 \\ 6 & -9 & 13 & -11 & 24 \\ -2 & 3 & 3 & 36 & -28 \end{bmatrix}$$

Find a $PA = LU$ factorization of A by applying the algorithm from class. You must show all of your steps.

Problem 2. Consider the $PA = LU$ factorization and the vector $\boldsymbol{b}$ where

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & 3 & -1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -2 & 5 & 7 & 5 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ -3 \\ -1 \\ 1 \end{bmatrix}$$

Use this factorization and the method of back substitution to find the general solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Eigenvalues

Name: ________________________________ NetID: ________________

Problem 1. The following two matrices A and B have the same eigenvalues

$$A = \left[\begin{array}{rr|rr} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ \hline 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{array}\right] \qquad B = \left[\begin{array}{rrr|r} -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ \hline 0 & 0 & 0 & 7 \end{array}\right]$$

The lines in these matrices are provided purely for aesthetic purposes.

(a) What are the geometric multiplicities of the eigenvalues of A?

(b) What are the geometric multiplicities of the eigenvalues of B?

Problem 2. Consider the matrix A given by

$$A = \begin{bmatrix} -107 & 315 & -420 \\ -28 & 82 & -112 \\ 7 & -21 & 26 \end{bmatrix}$$

The eigenvalues of A are $\operatorname{E-Vals}(A) = \{5, -2\}$.

(a) Find all solutions to $(-2 \cdot I_3 - A)\boldsymbol{x} = \mathbf{0}$.

(b) Find all solutions to $(5 \cdot I_3 - A)\boldsymbol{x} = \mathbf{0}$.

Problem 3. Suppose that λ is an eigenvalue of an $n \times n$ matrix A. Show that λ is also an eigenvalue of $A^\intercal$. *Hint.* Recall that every matrix M satisfies $\operatorname{rank}(M^\intercal) = \operatorname{rank}(M)$.

We learned in class that the eigenvalues of a triangular matrix are its diagonal entries. Since we can row reduce any matrix into triangular form, it can be tempting to associate this process with the concept of eigenvalues. The following exercise demonstrates that row reductions can not be relied on to calculate eigenvalues.

Problem 4. Consider the matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ -2 & -1 & 1 \end{bmatrix}$.

(a) Calculate $PA = LU$. What are the eigenvalues of U?

(b) Are any of the eigenvalues of U also eigenvalues of A? Clearly explain why or why not. Feel free do outsource any row reductions to a computer.

Null Spaces

Name: ____ NetID: ____________________

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} -8 & -12 & -60 & -48 \\ -1 & -2 & -9 & -7 \\ 3 & 5 & 24 & 19 \end{bmatrix}$$

(a) Let $\boldsymbol{v} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^\intercal$. Show that $\boldsymbol{v} \notin \text{Null}(A)$.

(b) Describe all vectors in $\text{Null}(A)$.

Problem 2. Consider the matrix A given by

$$A = \begin{bmatrix} -1 & -3 & 0 & -6 \\ -2 & -4 & 4 & -4 \\ -1 & -3 & 1 & -5 \\ 1 & 3 & -2 & 4 \end{bmatrix}$$

The scalar $\lambda = -1$ is an eigenvalue of A. Describe all of the vectors in the eigenspace $\mathcal{E}_A(-1)$.

Problem 3. Let A be an $n \times n$ matrix and define $M = A - 7 \cdot I_n$. Suppose that $\lambda = 3$ is an eigenvalue of A and that $\boldsymbol{v}$ is a corresponding eigenvector of A. Show that $\boldsymbol{v}$ is also an eigenvector of M and identify the corresponding eigenvalue.

One important feature of the Gramian $A^\intercal A$ of a matrix A is that it has the same null space as A (i.e. $\mathrm{Null}(A^\intercal A) = \mathrm{Null}(A)$). This next problem will walk you through the proof of this fact.

Problem 4. Let A be an $m \times n$ matrix.

(a) Suppose $\boldsymbol{v} \in \mathrm{Null}(A)$. Show that $\boldsymbol{v} \in \mathrm{Null}(A^\intercal A)$.

(b) Suppose $\boldsymbol{v} \in \mathrm{Null}(A^\intercal A)$. Show that $\boldsymbol{v} \in \mathrm{Null}(A)$ by showing that $\|A\boldsymbol{v}\|^2 = 0$.

Column Spaces

Name: ______________________________ NetID: ________________

Problem 1. Consider the matrix A and the vector $\boldsymbol{v}$ given by

$$A = \begin{bmatrix} -4 & 5 & -17 \\ -5 & 6 & -20 \\ 2 & 1 & -8 \\ 4 & -7 & 32 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 97 \\ -77 \\ -4 \\ -12 \end{bmatrix}$$

Determine if $\boldsymbol{v} \in \operatorname{Col}(A)$.

Problem 2. Suppose $EA = R$ where

$$E = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 2 & -3 & 3 \\ 1 & 0 & -2 & 1 \\ 1 & 3 & -1 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -5 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Use this factorization to determine if $\boldsymbol{v} = \begin{bmatrix} 1 & 1 & 4 & 7 \end{bmatrix}^\intercal$ is in $\operatorname{Col}(A)$.

Problem 3. Consider the matrix N and the vector $\boldsymbol{b}$ given by

$$N = \begin{bmatrix} -2 & 8 & -11 & -9 \\ 3 & -12 & 16 & 13 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

Suppose that A is a matrix with $\mathrm{Col}(A) = \mathrm{Null}(N)$.

(a) How many rows does A have? Explain.

(b) Determine if the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

The Four Fundamental Subspaces

Name: ____ NetID: ____________________

Problem 1. Suppose that A is a $4 \times n$ matrix whose rows satisfy the equations

$$\mathrm{Row}_1 - \mathrm{Row}_2 + \mathrm{Row}_3 + 2 \cdot \mathrm{Row}_4 = \boldsymbol{O} \qquad\qquad 3 \cdot \mathrm{Row}_1 - 2 \cdot \mathrm{Row}_2 + \mathrm{Row}_4 = \boldsymbol{O}$$

Find two nonparallel vectors in $\mathrm{Null}(A^\intercal)$.

Problem 2. Let $V = \{\begin{bmatrix} x & y \end{bmatrix}^\intercal \in \mathbb{R}^2 \mid xy = 0\}$. Show that V is *not* a vector space.

Hint. Note that $\begin{bmatrix} 1 & 0 \end{bmatrix}^\intercal \in V$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}^\intercal \in V$. Is every linear combination of these two vectors also in V?

Problem 3. Consider the following conditions

$$\begin{bmatrix}1 & -1 & 3 & 5\end{bmatrix}^\intercal \in \operatorname{Col}(A^\intercal) \qquad \begin{bmatrix}2 & 1 & 0 & 1\end{bmatrix}^\intercal \in \operatorname{Col}(A^\intercal) \qquad \begin{bmatrix}2 & -1 & 1\end{bmatrix}^\intercal \in \operatorname{Null}(A^\intercal)$$

Construct a matrix A with these properties and use a computer to find $\operatorname{rref}(A)$. *Hint.* Start by drawing the picture of the four fundamental subspaces and indicate where each of these three vectors belong in the picture.

Problem 4. Construct a matrix B such that $\operatorname{Null}(B) = \operatorname{Span}\{\begin{bmatrix}1 & -4 & 5 & -6\end{bmatrix}^\intercal, \begin{bmatrix}-7 & 29 & -30 & 39\end{bmatrix}^\intercal\}$. Then, use a computer to find $\operatorname{rref}(B)$.

Hint. This is the same as asking for $\operatorname{Col}(A) = \operatorname{Null}(B)$ for which A?

Linear Independence

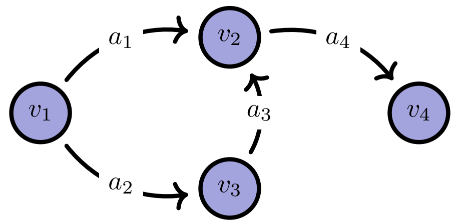
Name: ______________________________ NetID: ________________

Problem 1. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbb{R}^n$ and let A be an $m \times n$ matrix.

(a) Suppose that $\{A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3\}$ is linearly independent. Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

(b) Suppose that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent and that A has a left-inverse L (meaning $LA = I_n$). Show that $\{A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3\}$ is linearly independent.

Problem 2. Let A be the incidence matrix of the digraph given by



(a) Are the columns of A linearly independent? Explain why or why not.

(b) Suppose we remove a_4 from the digraph, and let A' be the new incidence matrix. Are the columns of A' linearly independent? Explain why or why not.

(c) Suppose in part (b) we had removed a_3 instead of a_4 and let A' be the new incidence matrix. Are the columns of A' linearly independent? Explain why or why not.

Bases

Name: ____ NetID: ____________________

Problem 1. Consider the matrices A and S given by

$$A = \begin{bmatrix} -40 & 50 \\ -25 & 35 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & -12 \\ -12 & -6 \end{bmatrix}$$

These matrices have the same eigenvalues $\text{E-Vals}(A) = \text{E-Vals}(S) = \{-15, 10\}$.

(a) Find bases of $\mathcal{E}_A(-15)$ and $\mathcal{E}_A(10)$. Then take the inner-product of each basis vector of $\mathcal{E}_A(-15)$ and $\mathcal{E}_A(10)$. What is this inner-product?

(b) Find bases of $\mathcal{E}_S(-15)$ and $\mathcal{E}_S(10)$. Then take the inner-product of each basis vector of $\mathcal{E}_S(-15)$ and $\mathcal{E}_S(10)$. What is this inner-product?

Problem 2. Consider the data

$$A = \begin{bmatrix} 1 & 16 & -18 & -6 \\ -8 & 3 & -8 & -24 \\ -9 & 8 & -14 & -27 \\ 1 & -8 & 9 & 6 \end{bmatrix} \qquad\qquad \text{E-Vals}(A) = \{-5, 3\}$$

(a) Find a basis of the eigenspace $\mathcal{E}_A(-5)$.

(b) Find a basis of the eigenspace $\mathcal{E}_A(3)$.

(c) Count the number of basis vectors you found for $\mathcal{E}_A(-5)$ and $\mathcal{E}_A(3)$. How do these counts compare to the geometric multiplicities $\operatorname{gm}_A(-5)$ and $\operatorname{gm}_A(3)$?

Dimension

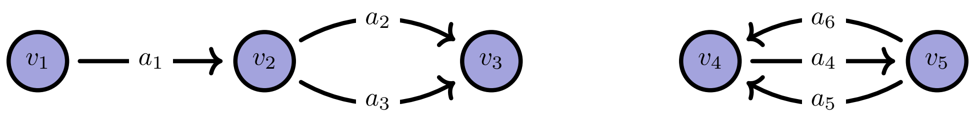
Name: ______________________________ NetID: ____________________

Problem 1. If possible, construct a matrix A with the following properties

$$\begin{bmatrix}-5 & 12 & -41\end{bmatrix}^{\intercal}, \begin{bmatrix}-3 & 7 & -24\end{bmatrix}^{\intercal} \in \operatorname{Null}(A) \qquad \begin{bmatrix}1 & 0 & 4 & 0\end{bmatrix}^{\intercal}, \begin{bmatrix}0 & 1 & 5 & -7\end{bmatrix}^{\intercal} \in \operatorname{Col}(A)$$

If this is not possible, then explain why.

Problem 2. Let A be the incidence matrix of the digraph G given by



(a) Draw the picture of the four fundamental subspaces of A. Label each space with its dimension.

(b) Is it possible to add three nodes and nine arrows to G to construct a new digraph G' whose circuit rank is 13? Rigorously argue why or why not using concepts covered in class.

Problem 3. Consider $A = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 3 & -6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 10 & -15 & 0 \\ 0 & 0 & 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}}$. Do not multiply to find A!

(a) Draw the picture of the four fundamental subspaces of A and include their dimensions.

(b) Find the pivot columns of A. These vectors form a basis of _______________.

(c) Find a basis of $\mathrm{Col}(A^\intercal)$.

(d) Find a basis of $\mathrm{Null}(A)$.

(e) List every vector in $\mathrm{Null}(A^\intercal)$.

Orthogonality

Name: ______________________________ NetID: _______________

Problem 1. Let S be the vector space given by

$$S = \operatorname{Span}\{\begin{bmatrix}1 & 2 & -6 & -1 & -3\end{bmatrix}^\intercal, \begin{bmatrix}1 & 3 & -8 & -2 & -4\end{bmatrix}^\intercal\}$$

Find a basis of $S^\perp$. Put your basis vectors into the rows of a matrix B and use a computer to find $\operatorname{rref}(B)$.

Problem 2. Let P be the vector space of solutions to the equations

$$-9\,x_1 + 47\,x_2 - 38\,x_3 - 112\,x_4 + 74\,x_5 = 0 \qquad -5\,x_1 + 26\,x_2 - 21\,x_3 - 62\,x_4 + 41\,x_5 = 0$$

Find a basis of $P^\perp$. Put your basis vectors into the rows of a matrix B and use a computer to find $\mathrm{rref}(B)$.

Problem 3. Let A be the incidence matrix of the digraph G given by

Let $\boldsymbol{b} = \begin{bmatrix} 5 & -3 & -1 & -1 \end{bmatrix}^\intercal$. Determine if the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent *without finding* A.

Hint. We can find a basis of $\operatorname{Null}(A^\intercal)$ without finding A. How would this basis help in determining if $A\boldsymbol{x} = \boldsymbol{b}$ is consistent?

Projections

Name: ______________________ NetID: ______________

Problem 1. Suppose that an $n \times n$ matrix P is *symmetric* ($P^\intercal = P$) and *idempotent* ($P^2 = P$).

(a) Show that the matrix $I_n - P$ is symmetric.

(b) Show that the matrix $I_n - P$ is idempotent.

(c) Show that every vector $\boldsymbol{v} \in \mathbb{R}^n$ satisfies the equation

$$\langle P\boldsymbol{v}, (I_n - P)\boldsymbol{v}\rangle = 0$$

Hint. Start with the adjoint formula.

Problem 2. Let P be the projection matrix onto a vector space $V \subset \mathbb{R}^n$ where $\dim(V) = d$. Show that $\operatorname{trace}(P) = d$.

Hint. The projection formula from class says that $P = X(X^\intercal X)^{-1}X^\intercal$ where X is $?\times?$. Also, recall the general formula for trace $\operatorname{trace}(MN) = \operatorname{trace}(NM)$ when MN and NM are defined.

Problem 3. Consider the matrix Q (whose columns are independent) given by

$$Q = \frac{1}{\sqrt{17}}\begin{bmatrix} 2 & 2 & 0 \\ -2 & 2 & 3 \\ -3 & 0 & -2 \\ 0 & 3 & -2 \end{bmatrix}$$

This matrix is very convenient to work with because each column of Q is a unit vector orthogonal to all other columns of Q. **Do not ignore the factor of** $1/\sqrt{17}$ **used to define this matrix!**

(a) Calculate the Gramian of Q. The information given above allows you to do with without doing any arithmetic.

(b) Find the projection matrix P onto $\operatorname{Col}(Q)$ *without using a computer.*

Problem 4. Let $V \subset \mathbb{R}^5$ be the vector space of solutions to the equation

$$x_1 - x_2 - 3\,x_3 - 2\,x_4 + 2\,x_5 = 0$$

(a) Find a matrix A such that $\mathrm{Null}(A) = V$. Draw the picture of all four fundamental subspaces of A (including their dimensions).

(b) Find the projection matrix P_V *without using a computer.*

Hint. This is a difficult direct computation. The picture from part (a) helps make this problem easier!

Least Squares

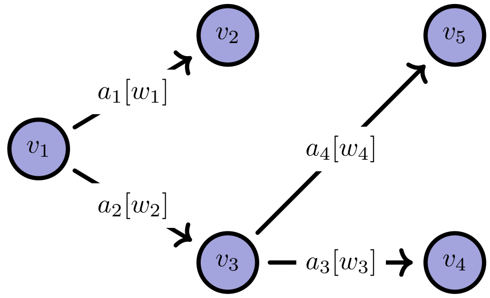
Name: ____ NetID: ____________________

Problem 1. Consider the data points

$$\{(0,0),(1,8),(3,8),(4,20)\}$$

Find the *parabola of best fit* $\widehat{f}(t) = \widehat{a}_0 + \widehat{a}_1\, t + \widehat{a}_2\, t^2$ to this data.

Problem 2. Consider the weighted digraph given by



Let A be the incidence matrix and let $\boldsymbol{b} = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \end{bmatrix}^\intercal$.

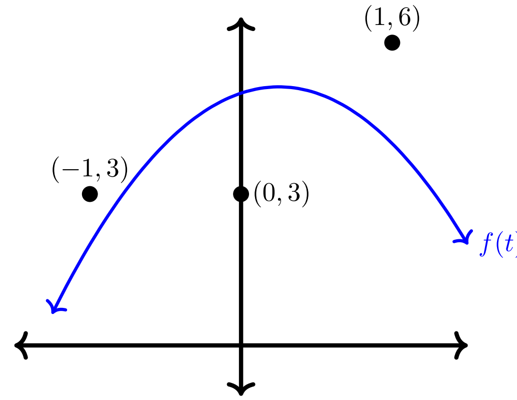
(a) Show that $A\boldsymbol{w} = \boldsymbol{b}$ is inconsistent *without doing any row-reductions or any computer calculations.*
Hint. The consistency of $A\boldsymbol{w} = \boldsymbol{b}$ hinges on whether or not $\boldsymbol{b} \in \operatorname{Col}(A) = \operatorname{Null}(A^\intercal)^\perp$. Start by finding a basis of $\operatorname{Null}(A^\intercal)$.

(b) Find the least squares approximate solution $\widehat{\boldsymbol{w}}$ to $A\boldsymbol{w} = \boldsymbol{b}$. Here, you may use a computer but you must explain your reasoning.

Problem 3. This figure depicts the result of using the least-squares method to fit a quadratic of the form

$$f(t) = c_1 \cdot (t^2 + t - 4) + c_2 \cdot (t^2 - 2\,t - 1)$$

to three data points. Find $f(t)$.



$A = QR$ Factorizations

Name: ____ NetID: ____________________

Problem 1. Suppose that A is a 2021×3 matrix whose columns are mutually orthogonal and have lengths 10, 11, and 12 respectively. Find $A^{\intercal}A$.

Problem 2. Suppose that Q_1 and Q_2 have orthonormal columns. Show that $Q = Q_1Q_2$ also has orthonormal columns.

Problem 3. Consider the data

$$\overset{A}{\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -4 \\ -1 & 1 \end{bmatrix}} = \overset{Q}{\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -1/6 \\ 1/2 & -5/6 \\ -1/2 & -1/6 \end{bmatrix}} \overset{R}{\begin{bmatrix} 2 & -3 \\ 0 & 3 \end{bmatrix}} \qquad\qquad \boldsymbol{b} = \begin{bmatrix} 4 \\ 2 \\ -6 \\ 4 \end{bmatrix}$$

(a) Use the QR-factorization to find the least-squares approximate solution $\widehat{\boldsymbol{x}}$ to $A\boldsymbol{x} = \boldsymbol{b}$.

(b) Use the QR-factorization to find the projection of $\boldsymbol{b}$ onto $\mathrm{Col}(A)$.

(c) Suppose that $\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4\}$ is an orthonormal basis of $\mathbb{R}^4$ where $\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ are the two columns of Q. Which (if any) of the four fundamental subspaces of A do $\{\boldsymbol{q}_3, \boldsymbol{q}_4\}$ belong to? Explain.

The Gram-Schmidt Algorithm

Name: ______________________________ NetID: ________________

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 2 & 4 & -6 \\ -2 & 0 & -1 \\ 0 & -3 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

Use the Gram-Schmidt algorithm to find a QR factorization of A. You can do calculations on a computer, but you must show every step of the algorithm to receive credit.

Problem 2. The Gram-Schmidt algorithm can be used as an alternative to row-reducing to determine the pivot and nonpivot columns of a matrix. Consider the matrix A given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & -1 & 3 \\ -1 & -3 & 6 & -5 \\ 1 & 3 & -6 & 3 \end{bmatrix}$$

Find the vectors $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_4\}$ according to the Gram-Schmidt algorithm. If any $\boldsymbol{w}_k$ is the zero vector, then do not use $\boldsymbol{w}_k$ to calculate the subsequent vectors. The kth column of A is a pivot column if and only if $\boldsymbol{w}_k \neq \boldsymbol{O}$!

Use this procedure to determine the pivot and nonpivot columns of A.

Determinants I

Name: ______________________________________ NetID: ____________________

Problem 1. Recall that a matrix A is called *idempotent* if $A^2 = A$. Show that every idempotent matrix A satisfies $\det(A) = 1$ or $\det(A) = 0$.

Problem 2. Consider two matrices A and B of the form

$$A = \begin{bmatrix} \mathrm{Row}_1 \\ \mathrm{Row}_2 \\ \mathrm{Row}_3 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} \mathrm{Row}_1 + \mathrm{Row}_2 + \mathrm{Row}_3 \\ \mathrm{Row}_1 + \mathrm{Row}_2 \\ \mathrm{Row}_1 \end{bmatrix}$$

Suppose $\det(A) = -11$. Find $\det(B)$.

Problem 3. Suppose that A is 5×5 with $\det(A) = 3$. Find $\det(-2 \cdot A)$. In general, what is $\det(c \cdot A)$ if A is $n \times n$?

Problem 4. A matrix S is called *skew-symmetric* if $S^{\intercal} = -S$. Suppose that S is 2023×2023 and skew-symmetric. Find $\det(S)$.

Determinants II

Name: ________________________________ NetID: ____________________

Problem 1. Consider the matrix X given by

$$X = \begin{bmatrix} t-5 & -2 & -4 \\ 8 & t+2 & 10 \\ 4 & 1 & t+5 \end{bmatrix}$$

Find all values of t for which X is singular.

Problem 2. Consider the system of equations given by

$$\begin{array}{rcrcrcr} -8x & - & y & + & z & = & -1 \\ -6x & + & y & + & 4z & = & 2 \\ -x & + & y & - & 5z & = & -5 \end{array}$$

Use Cramer's rule to solve this system.

The *cross product* of $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ is the "symbolic determinant" given by

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{bmatrix} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} & -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} & \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \end{bmatrix}^\intercal$$

Note that $\boldsymbol{v} \times \boldsymbol{w}$ is a *vector*.

Problem 3. Let $\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w}$ where $\boldsymbol{v} = \begin{bmatrix} 3 & -7 & 5 \end{bmatrix}^\intercal$ and $\boldsymbol{w} = \begin{bmatrix} 5 & 4 & 2 \end{bmatrix}^\intercal$. Find $\boldsymbol{n}$ and calculate the inner products $\langle \boldsymbol{n}, \boldsymbol{v} \rangle$ and $\langle \boldsymbol{n}, \boldsymbol{w} \rangle$.

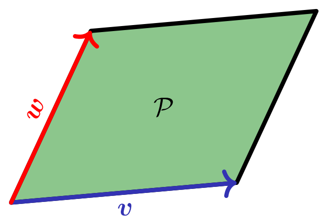
Problem 4. The *scalar triple product* states that any three vectors $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x} \in \mathbb{R}^3$ satisfy the equation

$$\langle \boldsymbol{x}, \boldsymbol{v} \times \boldsymbol{w} \rangle = \begin{vmatrix} x_1 & x_2 & x_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

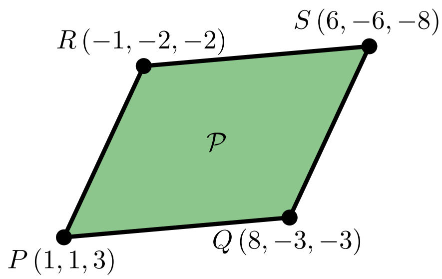
Use this equation to show that $\boldsymbol{v}$ is orthogonal to $\boldsymbol{v} \times \boldsymbol{w}$.

Hint. What is the rank of the matrix in this equation when $\boldsymbol{x} = \boldsymbol{v}$?

Problem 5. Consider the parallelogram $\mathcal{P}$ formed by two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$.



The area of this parallelogram is given by $\text{area}(\mathcal{P}) = \|\boldsymbol{v} \times \boldsymbol{w}\|$. Use this fact to calculate the area of the following parallelogram



Complex Numbers

Name: ______________________________ NetID: ________________

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 1 & i & i \\ i & -1 & i \end{bmatrix}$$

(a) Use elementary row operations to calculate $\operatorname{rref}(A)$ *without a computer*.

(b) Find a unit vector $\boldsymbol{u}$ in $\operatorname{Null}(A)$.

Problem 2. Suppose that $A^* = -A$. Show that $H = iA$ is Hermitian.

Problem 3. Suppose that $\boldsymbol{u} \in \mathbb{C}^n$ is a unit vector and define $Q = I_n - 2\,\boldsymbol{u}\boldsymbol{u}^*$.

(a) Show that Q is *Hermitian.*

(b) Show that Q is *unitary.*

Polynomial Algebra

Name: ______________________________ NetID: ________________

Problem 1. Let r_1, r_2, and r_3 be the three (not necessarily distinct) roots of the cubic

$$f(t) = t^3 + c_2\,t^2 + c_1\,t + c_0$$

Prove that $c_1 = r_1 \cdot r_2 + r_1 \cdot r_3 + r_2 \cdot r_3$.

Problem 2. Consider the polynomial $f(t)$ given by

$$f(t) = -67\,t^9 - 3\,t^8 + 4\,t^7 - 9\,t^6 + t^5 - t^4 - 2\,t^3 + t^2 + t + 4$$

This polynomial has nine distinct roots. Find the sum and product of these roots.

Problem 3. Factor $f(t) = t^7 - t^6 + t^5$ according to the Fundamental Theorem of Algebra.

Problem 4. Consider the monic polynomial $f(t)$ given by

$$f(t) = t^5 + 2\,t^4 - 2\,t^2 - 5\,t - 2$$

This polynomial has only one integer root r. Find r. *Hint.* We know that $f(r) = 0$. This tells us that r must divide what number?

Problem 5. Consider the polynomial $f(t) = (t-5)^3(t-7)^2$. Find the smallest value of m for which $f^{(m)}(5) \neq 0$ (recall that $f^{(m)}$ is the mth derivative of f, where $f^{(0)} = f$).

Problem 6. Consider again the polynomial $f(t) = (t-5)^3(t-7)^2$. Find the smallest value of m for which $f^{(m)}(7) \neq 0$.

Problem 7. The previous two problems demonstrate how we can use calculus to calculate the multiplicity of a root of a polynomial. The general method for doing this is encoded in the following theorem.

Theorem. *The multiplicity of a root r of a polynomial $f(t)$ is the smallest value m such that $f^{(m)}(r) \neq 0$.*

Note that $r = 1$ is a root of the polynomial $f(t)$ given by

$$f(t) = t^6 - 2\,t^5 - t^4 + 6\,t^3 - 7\,t^2 + 4\,t - 1$$

Use the above theorem to calculate the multiplicity of $r = 1$ as a root of $f(t)$.

The Characteristic Polynomial

Name: ________________________________ NetID: ________________

Problem 1. Consider the matrix A given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & y & z \end{bmatrix}$$

Since A is 3×3, the characteristic polynomial of A is of the form $\chi_A(t) = t^3 + c_2\, t^2 + c_1\, t + c_0$.

(a) Find c_0, c_1, and c_2.

(b) Suppose that $\text{E-Vals}(A) = \{-3, 5\}$ with $\text{am}_A(-3) = 1$ and $\text{am}_A(5) = 2$. Find x, y, and z.

Problem 2. Consider the matrix A given by

$$A = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}$$

Find all eigenvalues and bases for all eigenspaces of A *without using a computer*.

The next two problems will make reference to the matrix $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ for $0 \leq \theta < 2\pi$.

Problem 3. Find all eigenvalues of $R_{\pi/4}$.

Problem 4. Find all θ for which R_θ has exclusively real eigenvalues.

Problem 5. Consider the matrix A given by

$$A = \begin{bmatrix} -40 & -90 & 180 \\ 10 & 25 & -40 \\ -5 & -10 & 25 \end{bmatrix}$$

Find all eigenvalues of A *without using a computer.*

Problem 6. Suppose that A is an $n \times n$ matrix whose characteristic polynomial is

$$\chi_A(t) = t^7 - 2\,t^6 + 2\,t^5 - 2\,t^4 + 2\,t^3 - 3\,t^2 + 3\,t - 1$$

(a) $n =$ ________, trace$(A) =$ ________, and det$(A) =$ ________

(b) Is A invertible? Explain why or why not.

(c) Show that $\lambda = 1$ is an eigenvalue of A and calculate $\mathrm{am}_A(1)$.
Hint. Consider the last problem from "Polynomial Algebra".

Diagonalization

Name: ________________________________ NetID: ________________

Problem 1. Suppose that A is similar to B. Show that $\chi_A(t) = \chi_B(t)$.

Problem 2. Suppose that $A \sim B$ so $A = XBX^{-1}$ and consider $\boldsymbol{v} \in \mathcal{E}_B(\lambda)$. Show that $X\boldsymbol{v} \in \mathcal{E}_A(\lambda)$.

Problem 3. There is only one matrix A whose eigenspaces are

$$\mathcal{E}_A(-3) = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\} \qquad \mathcal{E}_A(5) = \operatorname{Span}\left\{\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right\} \qquad \mathcal{E}_A(11) = \operatorname{Span}\left\{\begin{bmatrix}-2\\1\\0\\-2\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\0\end{bmatrix}\right\}$$

Find A. You may do any calculations on a computer, but you must show your steps and explain your reasoning.

Problem 4. Consider the matrix A given by

$$A = \begin{bmatrix} -47 & 12 & -204 & -24 \\ 20 & -7 & 88 & 4 \\ 12 & -4 & 53 & 4 \\ -4 & 4 & -20 & 5 \end{bmatrix}$$

Show that A is diagonalizable and find X and D so that $A = XDX^{-1}$. You are free to outsource annoying calculations to a computer (like factoring $\chi_A(t)$) but you must show your steps to receive credit.

Problem 5. Let $A = \begin{bmatrix} -3 & 6 \\ -3 & 6 \end{bmatrix}$. Write A as $A = XDX^{-1}$ where X is invertible and D is diagonal. If no such factorization exists, then explain why.

Problem 6. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Write A as $A = XDX^{-1}$ where X is invertible and D is diagonal. If no such factorization exists, then explain why.

Matrix Exponentials

Name: ________________________________ NetID: ____________________

Problem 1. Suppose that A is idempotent. Prove that $\exp(At) = I + (e^t - 1)A$.

Problem 2. Calculate $\exp(At)$ for $A = \begin{bmatrix} -26 & -39 \\ 18 & 27 \end{bmatrix}$ by using the result from the previous problem.

Problem 3. Suppose that P is projection in $\mathbb{R}^2$ onto the line $y = x$. Calculate $\lim_{t\to\infty} \boldsymbol{u}(t)$ where $\boldsymbol{u}(t)$ is the solution $\boldsymbol{u}(t)$ to the initial value problem $\boldsymbol{u}' = -P\boldsymbol{u}$ with $\boldsymbol{u}(0) = \begin{bmatrix} 3 & 1 \end{bmatrix}^\intercal$.

Problem 4. A rabbit population r shows fast growth but is vulnerable to a wolf population w. This predator-prey model is exhibited by the initial value problem

$$r' = 6\,r - 2\,w \qquad w' = 2\,r + w \qquad r(0) = 30 \qquad w(0) = 30$$

What are the populations at time t? After a long time, what is the ratio of rabbits to wolves?

The Spectral Theorem

Name: ______________________________ NetID: ________________

Problem 1. Consider the real-symmetric matrix S given by

$$S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

Find *all* unitary matrices U that diagonalize S.

Problem 2. Consider the real-symmetric matrix S given by

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

Find a spectral factorization of S *without using a computer.*

Problem 3. A matrix S is *skew-Hermitian* if $S^* = -S$. Suppose that λ is an eigenvalue of a skew-Hermitian matrix S. Show that $\lambda = -\overline{\lambda}$.

Hint. Follow the proof of Part I of the spectral theorem.

Problem 4. The real symmetric matrix S below has exactly three eigenvalues.

$$S = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 4 & 5 & 4 & 0 \\ -3 & 4 & 4 & -1 \\ 1 & 0 & -1 & 8 \end{bmatrix} \quad \mathcal{E}_S(\lambda_1) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \mathcal{E}_S(\lambda_2) = \operatorname{Span}\{\boldsymbol{u}_2\} \quad \mathcal{E}_S(9) = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right\}$$

(a) Find the correct value of λ_1.

(b) Find the correct value of λ_2.

(c) There are only two valid unit vectors that $\boldsymbol{u}_2$ could equal. Find these two unit vectors. You may need to do some row reductions on a computer.

(d) Find a spectral factorization $S = UDU^{\intercal}$.

Definiteness

Name: ______________________________ NetID: ______________

Problem 1. Let $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$ where

$$S = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}$$

Complete the square to write $q(\boldsymbol{x}) = \lambda_1\, y_1^2 + \lambda_2\, y_2^2$ and determine the definiteness of S. Write each of y_1 and y_2 in terms of x_1 and x_2.

Problem 2. Suppose that S_1 and S_2 are positive definite matrices. Show that $S_1 + S_2$ is also positive definite.

Hint. Consider $q(\boldsymbol{x}) = \langle \boldsymbol{x}, (S_1 + S_2)\boldsymbol{x} \rangle$.

Problem 3. Suppose d_1, d_2, and d_3 are real numbers and consider the matrix S given by

$$S = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & -9 & 1 \end{bmatrix}} \overset{D}{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}} \overset{L^\intercal}{\begin{bmatrix} 1 & -3 & 7 \\ 0 & 1 & -9 \\ 0 & 0 & 1 \end{bmatrix}}$$

Let $q(\boldsymbol{x})$ be the quadratic form $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$.

(a) Prove that S is real-symmetric *without calculating* S.

(b) It is possible to write $q(\boldsymbol{x}) = d_1\, y_1^2 + d_2\, y_2^2 + d_3\, y_3^2$. Find y_1, y_2, and y_3 in terms of x_1, x_2, and x_3.

(c) Under what condition is S positive definite? Clearly explain your condition.

Problem 4. Consider the quadratic form on $\mathbb{R}^4$ given by

$$q(\boldsymbol{x}) = \frac{2}{36}(5\,x_1 - 3\,x_2 - x_3 + x_4)^2 + \frac{1}{6}(x_1 + x_2 + 2\,x_3)^2 + \frac{-2}{12}(x_1 + x_2 - x_3 - 3\,x_4)^2 + \frac{-7}{18}(x_1 + 3\,x_2 - 2\,x_3 + 2\,x_4)^2$$

Let S be the real-symmetric matrix satisfying $\boldsymbol{q}(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x}\rangle$.

(a) Find a spectral factorization $S = UDU^{\mathsf{T}}$ of S without finding S and determine the definiteness of S.

(b) Show that $\exp(S)$ is real symmetric and determine the definiteness of $\exp(S)$.

Singular Value Decomposition

Name: ______________________________ NetID: ________________

Problem 1. Find a singular value decomposition of $A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \\ -1 & 0 \end{bmatrix}$ *without using a computer.*

Problem 2. Use your work from Problem 1 to find the rank one approximation of the matrix A from Problem 1 (you are graded on correctness, so make sure your work in Problem 1 is correct).

Problem 3. Suppose that $A^*A = VDV^*$ where D is real-diagonal with positive diagonal entries and V has orthonormal columns. Further suppose that $\Sigma = \sqrt{D}$ (so that $D = \Sigma^2$). Show that $U = AV\Sigma^{-1}$ has orthonormal columns.

Problem 4. Suppose that A is a matrix satisfying

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\2\end{bmatrix} \qquad\qquad A\begin{bmatrix}2\\-2\end{bmatrix} = \begin{bmatrix}2\\1\\-2\end{bmatrix}$$

Find a singular value decomposition of A.

Problem 5. Suppose that $A = QR$.

(a) Show that A and R have the same singular values.

(b) Suppose that $R = U_R \Sigma_R V_R^*$ is a singular value decomposition. Find a singular decomposition $A = U\Sigma V^*$.

Problem 6. Find the rank one approximation of $A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & -2 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

Linear Approximations

Problem 1. Let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{f}(x, y) = \begin{bmatrix} x^2 + \sin(x) \\ x(y-2) \\ -3\,xy + y^2 \end{bmatrix}$$

Use the local linearization of $\boldsymbol{f}$ at the point $P = (0, 1)$ to approximate $f\left(1/2, 1/2\right)$.

Problem 2. Let $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

Use the local linearization of $\boldsymbol{r}$ at $t = {}^1\!/_2\,\pi$ to approximate $\boldsymbol{r}({}^1\!/_2\,\pi + {}^1\!/_3)$.

Problem 3. Consider $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{F}(r, \theta, z) = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ z \end{bmatrix}$$

Show that $\det(D\boldsymbol{F}) = r$.

Problem 4. Consider $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{F}(\rho, \varphi, \theta) = \begin{bmatrix} \rho\cos{(\theta)}\sin{(\varphi)} \\ \rho\sin{(\theta)}\sin{(\varphi)} \\ \rho\cos{(\varphi)} \end{bmatrix}$$

Show that $\det(D\boldsymbol{F}) = \rho^2\sin(\varphi)$.

Problem 5. Let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\boldsymbol{f}(r, \theta) = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$$

Let $R_\theta = D\boldsymbol{f}$ evaluated at $r = 1$. The matrix R_θ is called *the counterclockwise rotation matrix in* $\mathbb{R}^2$ *by* θ. The matrix-vector product $R_\theta \boldsymbol{v}$ is the vector $\boldsymbol{v}$ rotated counterclockwise by the angle θ.

(a) Calculate the matrix R_θ.

(b) Let $\boldsymbol{v} = \begin{bmatrix} x & y \end{bmatrix}^\intercal$. Find the coordinates of the vector $\boldsymbol{v}$ rotated counterclockwise by an angle of θ (this vector will depend on the coordinates x, y, and the angle θ.

(c) Find the coordinates of the vector $\boldsymbol{v} = \begin{bmatrix} 5/2\sqrt{3} + 3/2 \\ 3/2\sqrt{3} - 5/2 \end{bmatrix}$ rotated counterclockwise by $\theta = 5\pi/3$.

Cholesky Factorizations

Problem 1. Find the definiteness of $S = \begin{bmatrix} 4 & -2 & -2 & -20 \\ -2 & 1 & 1 & 10 \\ -2 & 1 & 17 & 14 \\ -20 & 10 & 14 & 105 \end{bmatrix}$.

Problem 2. Calculate $S = LDL^\intercal$ where $S = \begin{bmatrix} 2 & 12 & 10 \\ 12 & 67 & 105 \\ 10 & 105 & -351 \end{bmatrix}$.

Problem 3. Suppose that factoring $S = LDL^\intercal$ allows us to write the quadratic form $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$ as

$$q(\boldsymbol{x}) = 10\,(x_1 - 5\,x_2 + 2\,x_3)^2 - 11\,(x_2 - 6\,x_3)^2 - 5\,x_3^2$$

Find L and D and determine the definitess of S.

Problem 4. Determine the definiteness of $S = \begin{bmatrix} 0 & 4 & -6 & 8 & 16 \\ 4 & -650 & 3 & 1941 & -1 \\ -6 & 3 & 8 & -144 & 16 \\ 8 & 1941 & -144 & 2 & 18 \\ 16 & -1 & 16 & 18 & 52 \end{bmatrix}$.

Problem 5. Find R the Cholesky factorization $S = R^\intercal R$ of $S = \begin{bmatrix} 9 & 15 & -6 \\ 15 & 29 & 8 \\ -6 & 8 & 206 \end{bmatrix}$.

The Hessian

Problem 1. Consider the point $P\,(17,-22,37)$ and suppose $f \in \mathscr{C}(\mathbb{R}^3)$ is a scalar field satisfying

$$Hf(P) = \begin{bmatrix} -4 & 2 & -2 \\ 2 & -2 & 3 \\ -2 & 3 & -8 \end{bmatrix}$$

Suppose we use the local linearization of f at P to estimate $f(17-1,-22+1,37+1)$. Do we expect this estimation to be an overestimate or an underestimate? Explain.

Problem 2. Consider the point $P\,(1,1,1)$ and the scalar field $f \in \mathscr{C}(\mathbb{R}^3)$ given by

$$f(x,y,z) = x^3 z + y^3 + x^2 z + z^3$$

(a) Use the local linearization of f at P to approximate $f(1 + {}^1\!/\!_4, 1 - {}^1\!/\!_3, 1 - {}^1\!/\!_5)$.

(b) Use the second degree Taylor polynomial of f at P to approximate $f(1 + {}^1\!/\!_4, 1 - {}^1\!/\!_3, 1 - {}^1\!/\!_5)$.

(c) Do you expect your linear approximation in part (a) to be an overestimate or undestimate?