Linear Approximations

Problem 1. Let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{f}(x, y) = \begin{bmatrix} x^2 + \sin(x) \\ x(y-2) \\ -3\,xy + y^2 \end{bmatrix}$$

Use the local linearization of $\boldsymbol{f}$ at the point $P = (0, 1)$ to approximate $f\left(1/2, 1/2\right)$.

Problem 2. Let $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

Use the local linearization of $\boldsymbol{r}$ at $t = {}^1\!/_2\,\pi$ to approximate $\boldsymbol{r}({}^1\!/_2\,\pi + {}^1\!/_3)$.

Problem 3. Consider $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{F}(r, \theta, z) = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ z \end{bmatrix}$$

Show that $\det(D\boldsymbol{F}) = r$.

Problem 4. Consider $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{F}(\rho, \varphi, \theta) = \begin{bmatrix} \rho\cos(\theta)\sin(\varphi) \\ \rho\sin(\theta)\sin(\varphi) \\ \rho\cos(\varphi) \end{bmatrix}$$

Show that $\det(D\boldsymbol{F}) = \rho^2 \sin(\varphi)$.

Problem 5. Let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\boldsymbol{f}(r, \theta) = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$$

Let $R_\theta = D\boldsymbol{f}$ evaluated at $r = 1$. The matrix R_θ is called *the counterclockwise rotation matrix in* $\mathbb{R}^2$ *by* θ. The matrix-vector product $R_\theta \boldsymbol{v}$ is the vector $\boldsymbol{v}$ rotated counterclockwise by the angle θ.

(a) Calculate the matrix R_θ.

(b) Let $\boldsymbol{v} = \begin{bmatrix} x & y \end{bmatrix}^\intercal$. Find the coordinates of the vector $\boldsymbol{v}$ rotated counterclockwise by an angle of θ (this vector will depend on the coordinates x, y, and the angle θ.

(c) Find the coordinates of the vector $\boldsymbol{v} = \begin{bmatrix} {}^5\!/_2\sqrt{3} + {}^3\!/_2 \\ {}^3\!/_2\sqrt{3} - {}^5\!/_2 \end{bmatrix}$ rotated counterclockwise by $\theta = {}^{5\pi}\!/_3$.