Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: ______________________________ *Unique ID:* ____________________

I have adhered to the Duke Community Standard in completing this exam.

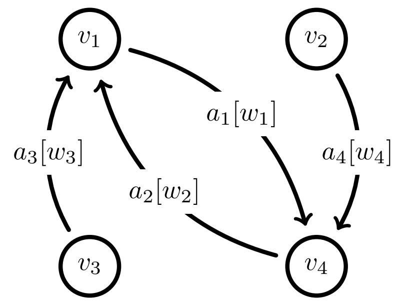
Signature: ______________________________

September 26, 2025

- There are 100 points and 7 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

Problem 1. The figure to the right depicts a weighted digraph G with unspecified weights w_1, w_2, w_3, and w_4.



(2 pts) (a) The Euler characteristic of G is ________.

(2 pts) (b) It is known that the following linear combination calculates the vector of net flow through the nodes of G.

$$5\cdot\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}+3\cdot\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}+7\cdot\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}+2\cdot\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$$

This information about net flow tells us that the weights in this digraph are:

$w_1 =$ ________ $w_2 =$ ________ $w_3 =$ ________ $w_4 =$ ________

Problem 2. Consider the matrix A and the vectors $\boldsymbol{v}$ and $\boldsymbol{b}$ in this product:

$$\overset{A}{\begin{bmatrix}18 & 11 & -14 & 1\\13 & 24 & 14 & 0\\-27 & 17 & 28 & 0\\19 & -22 & 21 & 1\\20 & 29 & 11 & 0\end{bmatrix}}\overset{\boldsymbol{v}}{\begin{bmatrix}1\\2\\-3\\0\end{bmatrix}}=\overset{\boldsymbol{b}}{\begin{bmatrix}82\\19\\-77\\-88\\45\end{bmatrix}}.$$

(2 pts) (a) The notation $\mathbb{R}^i \xrightarrow{A} \mathbb{R}^j$ makes sense for $i =$ ________ and $j =$ ________.

(3 pts) (b) Only one of the following statements accurately describes the geometric relationship between the rows of A and the vector $\boldsymbol{v}$. Select this statement.

- ○ Three rows of A are orthogonal to $\boldsymbol{v}$ and two rows of A are acute to $\boldsymbol{v}$.
- ○ Three rows of A are acute to $\boldsymbol{v}$ and two rows of A are obtuse to $\boldsymbol{v}$.
- ○ Three rows of A are obtuse to $\boldsymbol{v}$ and two rows of A are acute to $\boldsymbol{v}$.
- ○ Three rows of A are obtuse to $\boldsymbol{v}$ and two rows of A are orthogonal to $\boldsymbol{v}$.

(6 pts) (c) Calculate this matrix-vector product:

$$\begin{bmatrix}-14 & 1 & 11 & 18\\14 & 0 & 24 & 13\\28 & 0 & 17 & -27\\21 & 1 & -22 & 19\\11 & 0 & 29 & 20\end{bmatrix}\begin{bmatrix}-3\\10\\2\\1\end{bmatrix}=\begin{bmatrix}\\ \\ \\ \\ \end{bmatrix}$$

(12 pts) (d) Calculate the inner product $\langle A^\intercal A\boldsymbol{v}, \boldsymbol{w}\rangle$ where $\boldsymbol{w} = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit. Fill in the blank at the bottom of this page to make your answer clear.

$\langle A^\intercal A\boldsymbol{b}, \boldsymbol{w}\rangle =$ ________

Problem 3. Suppose that M is an $m \times n$ matrix, that $\boldsymbol{v} \in \mathbb{R}^m$, and that $\boldsymbol{w} \in \mathbb{R}^n$. Throughout this problem we will define the notation $\boldsymbol{v} \odot \boldsymbol{w}$ by the formula $\boldsymbol{v} \odot \boldsymbol{w} = \boldsymbol{v}^\intercal M \boldsymbol{w}$ (viewing $\boldsymbol{v}$ as an $m \times 1$ matrix and $\boldsymbol{w}$ as an $n \times 1$ matrix).

(3 pts) (a) Which of the following statements most accurately describes the type of mathematical object produced by calculating $\boldsymbol{v} \odot \boldsymbol{w}$?

○ a vector in $\mathbb{R}^m$ ○ a vector in $\mathbb{R}^n$ ○ an $m \times n$ matrix ○ an $n \times n$ matrix ○ a scalar

(5 pts) (b) Consider the case where $m = n$, where $M = c \cdot I_n$ for a scalar c, and where $\boldsymbol{w} = \boldsymbol{v}$. Only one of the following formulas for $\boldsymbol{v} \odot \boldsymbol{v}$ is guaranteed to be correct under these conditions. Select this formula.

○ $\boldsymbol{v} \odot \boldsymbol{v} = c \cdot \|\boldsymbol{v}\|^2$ ○ $\boldsymbol{v} \odot \boldsymbol{v} = c^2 \cdot \boldsymbol{v}$ ○ $\boldsymbol{v} \odot \boldsymbol{v} = c \cdot \boldsymbol{v}$ ○ $\boldsymbol{v} \odot \boldsymbol{v} = c^2 \cdot \|\boldsymbol{v}\|$ ○ $\boldsymbol{v} \odot \boldsymbol{v} = c^2 \cdot I_n$

(5 pts) (c) Consider the case where $m = n$, where M is invertible, and where $\boldsymbol{w} = M^{-1}\boldsymbol{v}$. Only one of the following formulas for $\boldsymbol{v} \odot \boldsymbol{w}$ is guaranteed to be correct under these conditions. Select this formula.

○ $\boldsymbol{v} \odot \boldsymbol{w} = \|\boldsymbol{v}\|$ ○ $\boldsymbol{v} \odot \boldsymbol{w} = \boldsymbol{v}^2$ ○ $\boldsymbol{v} \odot \boldsymbol{w} = \boldsymbol{v}$ ○ $\boldsymbol{v} \odot \boldsymbol{w} = \|\boldsymbol{v}\|^2$ ○ $\boldsymbol{v} \odot \boldsymbol{w} = I_n$

(12 pts) (d) Now, consider the matrix M and the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ given by

$$M = \begin{bmatrix} 17937 & -2772 & 34802 & -758 \\ -8045 & 1243 & -15609 & 340 \\ -10079 & 1558 & -19556 & 426 \\ -9035 & 1414 & -17547 & 385 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} -9 \\ 4 \\ 5 \\ 2 \end{bmatrix}$$

It is known that $\boldsymbol{w} \in \mathcal{E}_M(3)$. Use this information to calculate $\boldsymbol{v} \odot \boldsymbol{w}$. Clearly explain your reasoning to receive credit. Fill in the blank at the bottom of this page to make your answer clear.

$\boldsymbol{v} \odot \boldsymbol{w} =$ ________

Problem 4. The equation to the right depicts the calculation of the reduced row echelon form of a 5×4 matrix A. Note that $\operatorname{rref}(A)$ has been labeled as R and that the second and fourth columns of A are currently unknown and marked as $*$.

$$\operatorname{rref}\overbrace{\begin{bmatrix} 1 & * & 4 & * \\ 1 & * & 5 & * \\ 2 & * & 9 & * \\ -3 & * & -13 & * \\ 0 & * & 2 & * \end{bmatrix}}^{A} = \overbrace{\begin{bmatrix} 1 & 5 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}^{R}$$

(3 pts) (a) Both of the matrices $A^\intercal A$ and $R^\intercal R$ are ________ $\times$ ________ with rank ________.

(5 pts) (b) The $(1, 1)$ entry of $A^\intercal A$ is ________ and the $(3, 4)$ entry of $R^\intercal R$ is ________.

(4 pts) (c) The second column of A is $\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$ and the fourth column of A is $\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$

(12 pts) **Problem 5.** Suppose that A and B are two invertible matrices such that $\operatorname{trace}(A) = 3$ and $\operatorname{trace}(B) = 5$. The product of A and B is

$$AB = \begin{bmatrix} 9 & 1 & -6 & 9 \\ -4 & 0 & 3 & -6 \\ -6 & -1 & 4 & -7 \\ -6 & -1 & 4 & -6 \end{bmatrix}$$

Use this information to calculate $\operatorname{trace}(X)$ where $X = 2 \cdot B(A^{-1}B)^{-1} - (A + B^{-1})B$. Clearly explain your reasoning to receive credit. Fill in the blank at the bottom of this page to make your answers clear.

$\operatorname{trace}(X) =$ ________

(12 pts) **Problem 6.** We have learned two deterministic algorithms for row-reducing matrices. The "Gauß-Jordan" algorithm reduces a matrix to reduced row echelon form and the "$PA = LU$" algorithm reduces a matrix to row echelon form.

Fill in the blank next to each matrix below with the notation defined in class for the first elementary row operation called for by the indicated algorithm. (2pts each graded independently with no partial credit)

Gauß-Jordan

$$\begin{bmatrix} 71 & 56 & 93 & 35 \\ 58 & 27 & 86 & 88 \\ 14 & 23 & 39 & 39 \\ 1 & 12 & 92 & 58 \\ 60 & 87 & 85 & 87 \end{bmatrix}$$ ________________

$PA = LU$

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$ ________________

Gauß-Jordan

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$ ________________

$PA = LU$

$$\begin{bmatrix} 3 & 7 & 9 & 4 \\ 0 & 5 & 7 & 5 \\ 0 & 15 & 0 & 1 \end{bmatrix}$$ ________________

$PA = LU$

$$\begin{bmatrix} 3 & 7 & 9 & 4 & 1 \\ 0 & -5 & 3 & -4 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & -6 & 4 \end{bmatrix}$$ ________________

Gauß-Jordan

$$\begin{bmatrix} 1 & 5 & 0 & 9 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$ ________________

(12 pts) **Problem 7.** The data below depicts the result of row-reducing a matrix A to a matrix $\widetilde{R}$. The elementary matrices corresponding to the row-reductions used have been multiplied together to produce a factorization of the form $EA = \widetilde{R}$. The notation $\widetilde{R}$ has been used here because $\widetilde{R} \neq \operatorname{rref}(A)$ (although it should be noted that $\widetilde{R}$ is very close to reduced row echelon form).

$$\overset{E}{\begin{bmatrix} 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & -1 \\ 1 & -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 1 & -6 & -1 \\ 1 & -1 & -28 & 1 \\ 1 & 0 & -17 & 1 \\ 0 & 1 & 11 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}} = \overset{\widetilde{R}}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -17 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 11 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The data also depicts a vector $\boldsymbol{b}$ whose first two entries are unknown and marked as b_1 and b_2.

Without doing any row-reductions, use the $EA = \widetilde{R}$ factorization to determine the values of b_1 and b_2 that make the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent. Use these values of b_1 and b_2 to express the general solution to $A\boldsymbol{x} = \boldsymbol{b}$ as $\boldsymbol{x} = \boldsymbol{x}_p + c_1 \cdot \boldsymbol{x}_1$. Clearly explain your reasoning to receive credit. Fill in the blanks at the bottom of this page to make your answers clear.

The system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for $b_1 =$ _______ and $b_2 =$ _______ and $\boldsymbol{x} = \overset{\boldsymbol{x}_p}{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}} + c_1 \cdot \overset{\boldsymbol{x}_1}{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}}$