Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: Solutions

Unique ID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

October 24, 2025

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

Problem 1. The data below depicts the equation $E(\lambda \cdot I_5 - A) = R$ where A is an unspecified 5×5 matrix, λ is an unspecified scalar quantity, E is nonsingular, and $R = \operatorname{rref}(\lambda \cdot I_5 - A)$.

$$\overset{E}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix}} \begin{bmatrix} \\ \\ \quad \lambda \cdot I_5 - A \quad \\ \\ \\ \end{bmatrix} = \overset{R}{\begin{bmatrix} 1 & -5 & 0 & -4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

(2 pts) (*a*) λ is an eigenvalue of A with $\operatorname{gm}_A(\lambda) = \underline{\quad 2 \quad}$.

(4 pts) (*b*) Only one of the following statements accurately applies to the eigenvectors of A corresponding to the eigenvalue λ. Select this statement.

- ○ A has exactly one eigenvector corresponding to the eigenvalue λ.
- ○ A has exactly two eigenvectors corresponding to the eigenvalue λ.
- ○ A has infinitely many eigenvectors corresponding to the eigenvalue λ, but these eigenvectors are all multiples of each other.
- √ It is possible to find two linearly independent eigenvectors of A corresponding to the eigenvalue λ but it is impossible to find three linearly independent eigenvectors of A corresponding to the eigenvalue λ.
- ○ None of the above.

(4 pts) (*c*) Only one of the following vectors is an eigenvector of A corresponding to the eigenvalue λ. Select this vector.

○ $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ ○ $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\boldsymbol{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ √ $\boldsymbol{v}_5 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}$

(4 pts) (*d*) Of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5$ defined as the options in part (*c*) of this problem, only one belongs to the *column space* of $\lambda \cdot I_5 - A$. Select this vector. ○ $\boldsymbol{v}_1$ √ $\boldsymbol{v}_2$ ○ $\boldsymbol{v}_3$ ○ $\boldsymbol{v}_4$ ○ $\boldsymbol{v}_5$

(4 pts) (*e*) Let P be the projection matrix onto the vector space $V = \mathcal{E}_A(\lambda)$. The matrix $I_5 - P$ is then the projection matrix onto only one of the following vector spaces. Select this vector space.

○ $\mathcal{E}_A(1-\lambda)$ ○ The column space of A. ○ The row space of A.

○ The column space of $\lambda \cdot I_5 - A$. √ The row space of $\lambda \cdot I_5 - A$.

(10 pts) (*f*) Find a basis of $\mathcal{E}_A(\lambda)$. Clearly explain your reasoning to receive credit. List your basis vectors in the box at the bottom of this page for clarity.

Solution. By definition, $\mathcal{E}_A(\lambda) = \operatorname{Null}(\lambda \cdot I_5 - A)$. Our primary technique for finding bases of null spaces is to find the "pivot solutions" to the defining equation, which is $(\lambda \cdot I_5 - A)\boldsymbol{x} = \boldsymbol{O}$ here. The given $R = \operatorname{rref}(\lambda \cdot I_5 - A)$ gives us everything we need. The system $(\lambda \cdot I_5 - A)\boldsymbol{x} = \boldsymbol{O}$ has five variables x_1, x_2, x_3, x_4, x_5 where $x_2 = c_1$ and $x_4 = c_2$ are free. The relevant equations are

$$x_1 - 5\,c_1 - 4\,c_2 = 0 \qquad x_3 + 3\,c_2 = 0 \qquad x_5 = 0$$

Solving for the dependent in terms of the free yields the general solution as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5\,c_1 + 4\,c_2 \\ c_1 \\ -3\,c_2 \\ c_2 \\ 0 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

This gives our basis as $\mathcal{E}_A(\lambda) = \operatorname{Span}\left[\begin{matrix} 5 & 1 & 0 & 0 & 0 \end{matrix}\right]^\intercal, \left[\begin{matrix} 4 & 0 & -3 & 1 & 0 \end{matrix}\right]^\intercal$.

Problem 2. The data below depicts two 5×3 matrices X and Y and the matrix $M = X(Y^\intercal X)^{-1}Y^\intercal$.

$$X = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -3 \\ -2 & 13 & -40 \\ -2 & 12 & -41 \\ -3 & 15 & -45 \end{bmatrix} \qquad Y = \begin{bmatrix} 5 & -17 & 99 \\ 5 & -14 & 90 \\ -1 & 2 & -13 \\ -3 & 9 & -56 \\ 2 & -7 & 40 \end{bmatrix} \qquad M = \begin{bmatrix} -433 & 485 & 381 & -163 & -290 \\ -109 & 122 & 96 & -41 & -73 \\ 24 & -27 & -20 & 9 & 16 \\ -423 & 471 & 372 & -158 & -283 \\ 735 & -822 & -645 & 276 & 492 \end{bmatrix}$$

It is known that both X and Y have linearly independent columns.

(5 pts) (a) $\operatorname{nullity}(X) = \operatorname{nullity}(Y) =$ __0__ and $\dim \operatorname{Null}(X^\intercal) = \dim \operatorname{Null}(Y^\intercal) =$ __2__

(4 pts) (b) Let $\boldsymbol{v}$ be a vector. Which of the following vector spaces must $\boldsymbol{v}$ belong to in order to guarantee that $\boldsymbol{v} \in \operatorname{Null}(M)$?

○ The column space of X. ○ The column space of Y. ○ The left null space of X.

√ The left null space of Y. ○ The null space of X.

(10 pts) (c) Show that M is *idempotent*. To receive credit, your work must be neatly organized and easy to follow.

Solution. We are given that $M = X(Y^\intercal X)^{-1}Y^\intercal$. We wish to demonstrate that M is *idempotent*, which means we wish to show that $M^2 = M$.

To do so, note that

$$\begin{aligned} M^2 &= \left(X(Y^\intercal X)^{-1}Y^\intercal\right)\left(X(Y^\intercal X)^{-1}Y^\intercal\right) \\ &= X\underbrace{(Y^\intercal X)^{-1}(Y^\intercal X)}_{=I_3}(Y^\intercal X)^{-1}Y^\intercal \\ &= X(Y^\intercal X)^{-1}Y^\intercal \\ &= M \end{aligned}$$

So indeed M is idempotent as promised.

(10 pts) (d) Let $\boldsymbol{v}$ be any vector in the column space of X (recall that this means that $\boldsymbol{v} = X\boldsymbol{x}$ for some vector $\boldsymbol{x}$). Show that $\boldsymbol{v} \in \mathcal{E}_M(\lambda)$ and identify the correct value of λ. To receive credit, your work must be neatly organized and easy to follow. Record your value of λ in the blank below for clarity.

Solution. We are given that $\boldsymbol{v} = X\boldsymbol{x}$. We wish to demonstrate that $\boldsymbol{v} \in \mathcal{E}_M(\lambda)$, which means we wish to validate the equation $M\boldsymbol{v} = \lambda \cdot \boldsymbol{v}$. To do so, note that

$$\begin{aligned} M\boldsymbol{v} &= MX\boldsymbol{x} \\ &= X\underbrace{(Y^\intercal X)^{-1}Y^\intercal X}_{=I_3}\boldsymbol{x} \\ &= X\boldsymbol{x} \\ &= \boldsymbol{v} \\ &= 1 \cdot \boldsymbol{v} \end{aligned}$$

This demonstrates that $M\boldsymbol{v} = \lambda \cdot \boldsymbol{v}$ with $\lambda = 1$.

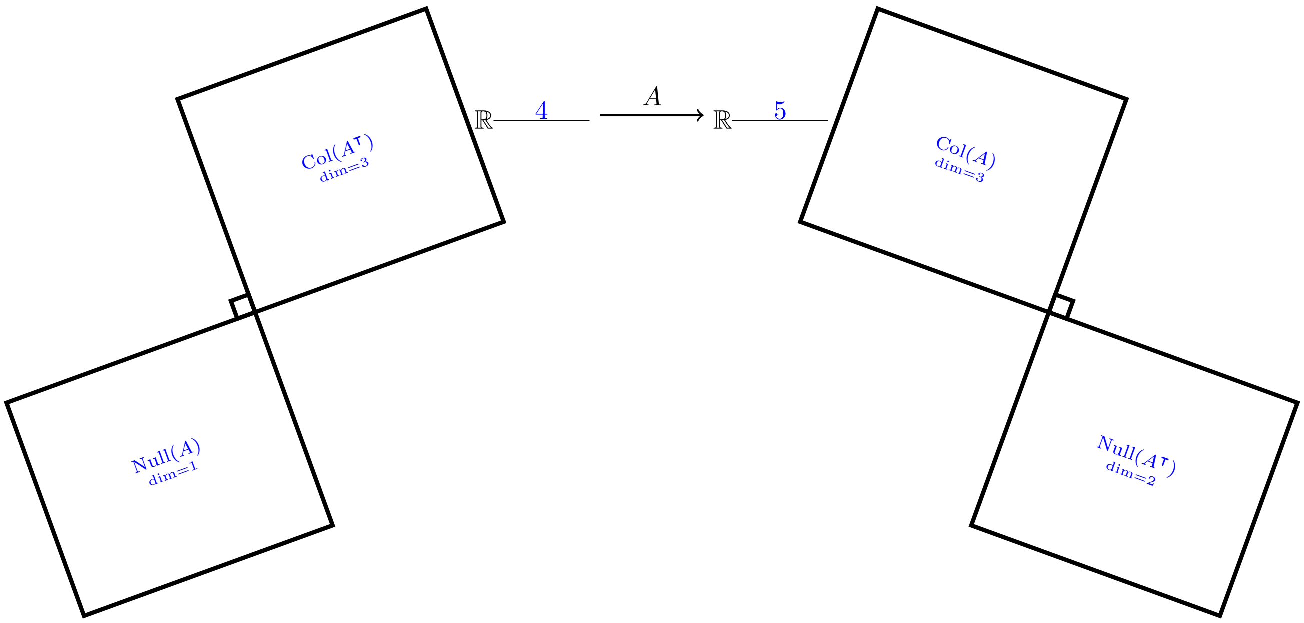
$\lambda =$ __1__

Problem 3. The data below depicts a 5×4 matrix A along with the projection matrix P onto the null space of A.

$$A = \begin{bmatrix} 1 & 1 & 8 & 10 \\ 4 & 5 & 37 & 43 \\ -4 & -9 & -56 & -54 \\ 5 & 2 & 30 & 46 \\ 3 & 2 & 24 & 32 \end{bmatrix} \qquad P = \frac{1}{22}\begin{bmatrix} 16 & -8 & 4 & -4 \\ -8 & 4 & -2 & 2 \\ 4 & -2 & 1 & -1 \\ -4 & 2 & -1 & 1 \end{bmatrix}$$

Do not ignore the factor of $1/22$ used to define P (for instance, the $(3,4)$ entry of P is $-1/22$)!

(10 pts) (a) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(5 pts) (b) Let $\boldsymbol{v}$ be any vector in Col(A) and let M be the 5×5 matrix whose first four columns are the same as the first four columns of A and whose fifth column is $\boldsymbol{v}$. Fill in each of the following blanks with a ">" sign, a "<" sign, or an "=" sign (2.5pts each).

$$\dim \operatorname{Col}(A) \underline{\quad = \quad} \dim \operatorname{Col}(M) \qquad \dim \operatorname{Null}(A) \underline{\quad < \quad} \dim \operatorname{Null}(M)$$

(10 pts) (c) Let $\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^\intercal$. Determine if it is possible to express $\boldsymbol{b}$ as a linear combination of the rows of A. Clearly explain your reasoning to receive credit.

Solution. This is the same as asking to determine whether or not $\boldsymbol{b}$ is in the *row space* of A, which is Col($A^\intercal$). The "from scratch" method of solving this problem would be to check the consistency of the system $A\boldsymbol{x} = \boldsymbol{b}$. The numbers in A are clearly too complicated to take this approach.

Instead, note that $\operatorname{Col}(A^\intercal) = \operatorname{Null}(A)^\perp$. Since we are given the projection matrix onto Null(A), we need only check whether or not $P\boldsymbol{b} = \boldsymbol{O}$. The relevant calculation here is

$$\frac{1}{22}\overset{P}{\begin{bmatrix} 16 & -8 & 4 & -4 \\ -8 & 4 & -2 & 2 \\ 4 & -2 & 1 & -1 \\ -4 & 2 & -1 & 1 \end{bmatrix}}\overset{\boldsymbol{b}}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This indicates that it is, in fact, possible to express $\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^\intercal$ as a linear combination of the rows of A. Note that it is equally valid to arrive at this conclusion by checking that $(I - P)\boldsymbol{b} = \boldsymbol{b}$ since $I - P$ is projection onto Col($A^\intercal$).

Problem 4. Let $f(t)$ be the curve defined by

$$f(t) = c_1 \cdot f_1(t) + c_2 \cdot f_2(t) + c_3 \cdot f_3(t)$$

where c_1, c_2, and c_3 are unknown scalars and $f_1(t)$, $f_2(t)$, and $f_3(t)$ are functions whose values at $t = 0, \pm 1, \pm 2, \pm 3$ are given in the table to the right of this paragraph (for example $f_1(3) = 1$ and $f_3(0) = -1$).

t	-3	-2	-1	0	1	2	3
$f_1(t)$	1	-1	-2	1	1	1	1
$f_2(t)$	1	2	-1	1	1	-1	1
$f_3(t)$	1	-1	-1	-1	2	-2	2

(10 pts) (*a*) It is impossible to find c_1, c_2, c_3 such that $f(t)$ fits all points in the dataset $\{(2, -7), (-2, 0), (0, 0), (-1, 0)\}$. Use the least squares technique to find $\widehat{f}(t) = \widehat{c}_1 \cdot f_1(t) + \widehat{c}_2 \cdot f_2(t) + \widehat{c}_3 \cdot f_3(t)$ that best fits this data. Record your values of $\widehat{c}_1$, $\widehat{c}_2$, and $\widehat{c}_3$ below for clarity.

Solution. An attempt to "perfectly" fit $f(t)$ to the data leads to the following equations.

$$\begin{array}{rclcrcrcrclcrcrcrcl}
f(2) &=& -7 &\leftrightarrow& c_1 \cdot f_1(2) &+& c_2 \cdot f_2(2) &+& c_3 \cdot f_3(2) &=& -7 &\leftrightarrow& c_1 &-& c_2 &-& 2\,c_3 &=& -7 \\
f(-2) &=& 0 &\leftrightarrow& c_1 \cdot f_1(-2) &+& c_2 \cdot f_2(-2) &+& c_3 \cdot f_3(-2) &=& 0 &\leftrightarrow& -c_1 &+& 2\,c_2 &-& c_3 &=& 0 \\
f(0) &=& 0 &\leftrightarrow& c_1 \cdot f_1(0) &+& c_2 \cdot f_2(0) &+& c_3 \cdot f_3(0) &=& 0 &\leftrightarrow& c_1 &+& c_2 &-& c_3 &=& 0 \\
f(-1) &=& 0 &\leftrightarrow& c_1 \cdot f_1(-1) &+& c_2 \cdot f_2(-1) &+& c_3 \cdot f_3(-1) &=& 0 &\leftrightarrow& -2\,c_1 &-& c_2 &-& c_3 &=& 0
\end{array}$$

The system of interest is then of the form $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & -1 \\ 1 & 1 & -1 \\ -2 & -1 & -1 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -7 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad A^\intercal A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \qquad A^\intercal \boldsymbol{b} = \begin{bmatrix} -7 \\ 7 \\ 14 \end{bmatrix}$$

The associated least squares problem is $A^\intercal A \widehat{\boldsymbol{x}} = A^\intercal \boldsymbol{b}$. Interestingly we find that $A^\intercal A = 7 \cdot I_3$, which easily inverts as $(A^\intercal A)^{-1} = \frac{1}{7} \cdot I_3$. Our least squares problem is then solved by

$$\widehat{\boldsymbol{x}} = (A^\intercal A)^{-1} A^\intercal \boldsymbol{b} = \frac{1}{7} A^\intercal \boldsymbol{b} = \frac{1}{7} \begin{bmatrix} -7 \\ 7 \\ 14 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Our least squares curve is thus $\widehat{f}(t) = -f_1(t) + f_2(t) + 2\, f_3(t)$.

$\widehat{c}_1 =$ <u>-1</u>, $\widehat{c}_2 =$ <u>1</u>, $\widehat{c}_3 =$ <u>2</u>

(8 pts) (*b*) It is also impossible to find c_1, c_2, and c_3 such that $f(t)$ fits all points in the dataset $\{(0, -2), (1, 0), (2, 1), (3, 2)\}$. The least squares version of $f(t)$ that best fits this dataset is $\widehat{f}(t) = f_1(t) - 2\, f_2(t) + f_3(t)$. Find the error E in this approximation. Record your value of E below for clarity.

Solution. An attempt to "perfectly" fit $f(t)$ to the data leads to the following equations.

$$\begin{array}{rclcrcrcrclcrcrcrcl}
f(0) &=& -2 &\leftrightarrow& c_1 \cdot f_1(0) &+& c_2 \cdot f_2(0) &+& c_3 \cdot f_3(0) &=& -2 &\leftrightarrow& c_1 &+& c_2 &-& c_3 &=& -2 \\
f(1) &=& 0 &\leftrightarrow& c_1 \cdot f_1(1) &+& c_2 \cdot f_2(1) &+& c_3 \cdot f_3(1) &=& 0 &\leftrightarrow& c_1 &+& c_2 &+& 2\,c_3 &=& 0 \\
f(2) &=& 1 &\leftrightarrow& c_1 \cdot f_1(2) &+& c_2 \cdot f_2(2) &+& c_3 \cdot f_3(2) &=& 1 &\leftrightarrow& c_1 &-& c_2 &-& 2\,c_3 &=& 1 \\
f(3) &=& 2 &\leftrightarrow& c_1 \cdot f_1(3) &+& c_2 \cdot f_2(3) &+& c_3 \cdot f_3(3) &=& 2 &\leftrightarrow& c_1 &+& c_2 &-& 2\,c_3 &=& 2
\end{array}$$

This is the system $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

We are told that the least squares curve is $\widehat{f}(t) = f_1(t) - 2\, f_2(t) + f_3(t)$, which means that the solution to our associated least squares problem $A^\intercal A \widehat{\boldsymbol{x}} = A^\intercal \boldsymbol{b}$ is $\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^\intercal$. The error in this approximation is then

$$E = \|\boldsymbol{b} - A\widehat{x}\|^2 = \|\begin{bmatrix} -2 & 0 & 1 & 2 \end{bmatrix}^\intercal - \begin{bmatrix} -2 & 1 & 1 & 1 \end{bmatrix}^\intercal\|^2 = \|\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}^\intercal\|^2 = 2$$

$E =$ <u>2</u>