Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: ______________________________ *Unique ID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________________

October 24, 2025

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. The data below depicts the equation $E(\lambda \cdot I_5 - A) = R$ where A is an unspecified 5×5 matrix, λ is an unspecified scalar quantity, E is nonsingular, and $R = \operatorname{rref}(\lambda \cdot I_5 - A)$.

$$\overset{E}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix}} \begin{bmatrix} & & & & \\ & & & & \\ & & \lambda \cdot I_5 - A & & \\ & & & & \\ & & & & \end{bmatrix} = \overset{R}{\begin{bmatrix} 1 & -5 & 0 & -4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

(2 pts) (a) λ is an eigenvalue of A with $\operatorname{gm}_A(\lambda) =$ ________.

(4 pts) (b) Only one of the following statements accurately applies to the eigenvectors of A corresponding to the eigenvalue λ. Select this statement.

- ○ A has exactly one eigenvector corresponding to the eigenvalue λ.
- ○ A has exactly two eigenvectors corresponding to the eigenvalue λ.
- ○ A has infinitely many eigenvectors corresponding to the eigenvalue λ, but these eigenvectors are all multiples of each other.
- ○ It is possible to find two linearly independent eigenvectors of A corresponding to the eigenvalue λ but it is impossible to find three linearly independent eigenvectors of A corresponding to the eigenvalue λ.
- ○ None of the above.

(4 pts) (c) Only one of the following vectors is an eigenvector of A corresponding to the eigenvalue λ. Select this vector.

○ $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ ○ $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\boldsymbol{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ○ $\boldsymbol{v}_5 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}$

(4 pts) (d) Of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5$ defined as the options in part (c) of this problem, only one belongs to the *column space* of $\lambda \cdot I_5 - A$. Select this vector. ○ $\boldsymbol{v}_1$ ○ $\boldsymbol{v}_2$ ○ $\boldsymbol{v}_3$ ○ $\boldsymbol{v}_4$ ○ $\boldsymbol{v}_5$

(4 pts) (e) Let P be the projection matrix onto the vector space $V = \mathcal{E}_A(\lambda)$. The matrix $I_5 - P$ is then the projection matrix onto only one of the following vector spaces. Select this vector space.

○ $\mathcal{E}_A(1 - \lambda)$ ○ The column space of A. ○ The row space of A.

○ The column space of $\lambda \cdot I_5 - A$. ○ The row space of $\lambda \cdot I_5 - A$.

(10 pts) (f) Find a basis of $\mathcal{E}_A(\lambda)$. Clearly explain your reasoning to receive credit. List your basis vectors in the box at the bottom of this page for clarity.

Basis of $\mathcal{E}_A(\lambda)$

Problem 2. The data below depicts two 5×3 matrices X and Y and the matrix $M = X(Y^\intercal X)^{-1}Y^\intercal$.

$$X = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -3 \\ -2 & 13 & -40 \\ -2 & 12 & -41 \\ -3 & 15 & -45 \end{bmatrix} \qquad Y = \begin{bmatrix} 5 & -17 & 99 \\ 5 & -14 & 90 \\ -1 & 2 & -13 \\ -3 & 9 & -56 \\ 2 & -7 & 40 \end{bmatrix} \qquad M = \begin{bmatrix} -433 & 485 & 381 & -163 & -290 \\ -109 & 122 & 96 & -41 & -73 \\ 24 & -27 & -20 & 9 & 16 \\ -423 & 471 & 372 & -158 & -283 \\ 735 & -822 & -645 & 276 & 492 \end{bmatrix}$$

It is known that both X and Y have linearly independent columns.

(5 pts) (a) $\operatorname{nullity}(X) = \operatorname{nullity}(Y) =$ _______ and $\dim \operatorname{Null}(X^\intercal) = \dim \operatorname{Null}(Y^\intercal) =$ _______

(4 pts) (b) Let $\boldsymbol{v}$ be a vector. Which of the following vector spaces must $\boldsymbol{v}$ belong to in order to guarantee that $\boldsymbol{v} \in \operatorname{Null}(M)$?

○ The column space of X. ○ The column space of Y. ○ The left null space of X.

○ The left null space of Y. ○ The null space of X.

(10 pts) (c) Show that M is *idempotent*. To receive credit, your work must be neatly organized and easy to follow.

(10 pts) (d) Let $\boldsymbol{v}$ be any vector in the column space of X (recall that this means that $\boldsymbol{v} = X\boldsymbol{x}$ for some vector $\boldsymbol{x}$). Show that $\boldsymbol{v} \in \mathcal{E}_M(\lambda)$ and identify the correct value of λ. To receive credit, your work must be neatly organized and easy to follow. Record your value of λ in the blank below for clarity.

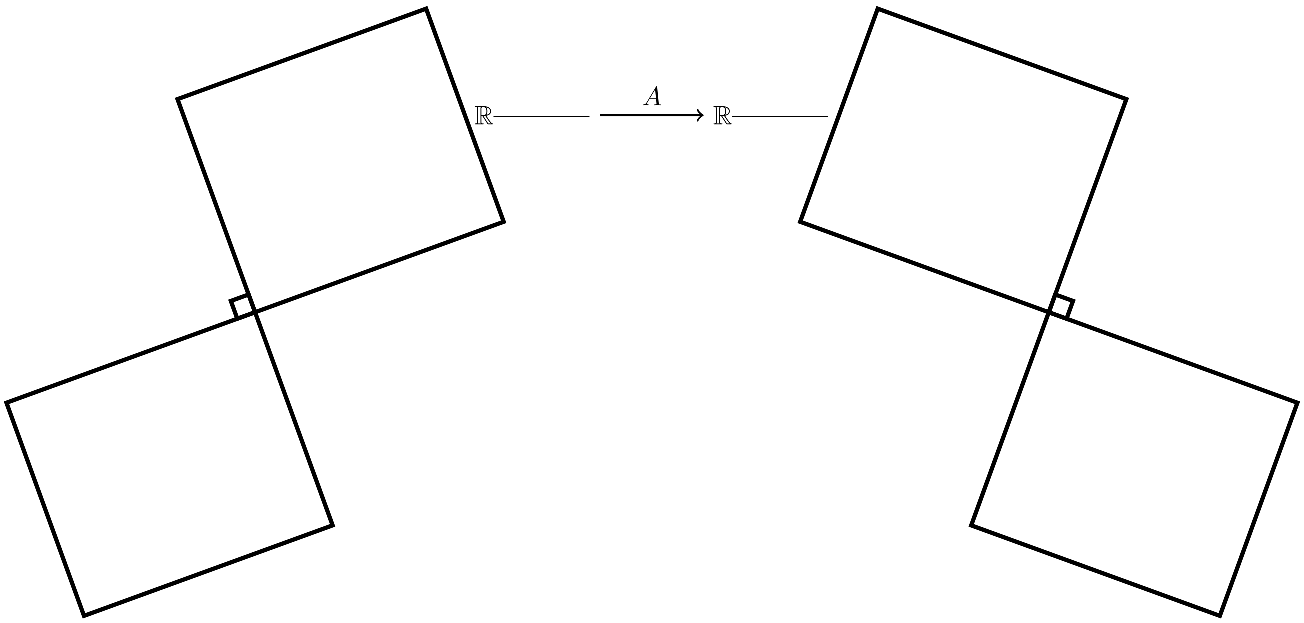
$\lambda =$ _______

Problem 3. The data below depicts a 5×4 matrix A along with the projection matrix P onto the null space of A.

$$A = \begin{bmatrix} 1 & 1 & 8 & 10 \\ 4 & 5 & 37 & 43 \\ -4 & -9 & -56 & -54 \\ 5 & 2 & 30 & 46 \\ 3 & 2 & 24 & 32 \end{bmatrix} \qquad P = \frac{1}{22}\begin{bmatrix} 16 & -8 & 4 & -4 \\ -8 & 4 & -2 & 2 \\ 4 & -2 & 1 & -1 \\ -4 & 2 & -1 & 1 \end{bmatrix}$$

Do not ignore the factor of $^1/_{22}$ used to define P (for instance, the $(3,4)$ entry of P is $-^1/_{22}$)!

(10 pts) (a) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(5 pts) (b) Let $\boldsymbol{v}$ be any vector in $\operatorname{Col}(A)$ and let M be the 5×5 matrix whose first four columns are the same as the first four columns of A and whose fifth column is $\boldsymbol{v}$. Fill in each of the following blanks with a ">" sign, a "<" sign, or an "=" sign (2.5pts each).

$\dim \operatorname{Col}(A)$________$\dim \operatorname{Col}(M)$ $\qquad$ $\dim \operatorname{Null}(A)$________$\dim \operatorname{Null}(M)$

(10 pts) (c) Let $\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^\intercal$. Determine if it is possible to express $\boldsymbol{b}$ as a linear combination of the rows of A. Clearly explain your reasoning to receive credit.

Problem 4. Let $f(t)$ be the curve defined by

$$f(t) = c_1 \cdot f_1(t) + c_2 \cdot f_2(t) + c_3 \cdot f_3(t)$$

where c_1, c_2, and c_3 are unknown scalars and $f_1(t)$, $f_2(t)$, and $f_3(t)$ are functions whose values at $t = 0, \pm 1, \pm 2, \pm 3$ are given in the table to the right of this paragraph (for example $f_1(3) = 1$ and $f_3(0) = -1$).

t	-3	-2	-1	0	1	2	3
$f_1(t)$	1	-1	-2	1	1	1	1
$f_2(t)$	1	2	-1	1	1	-1	1
$f_3(t)$	1	-1	-1	-1	2	-2	2

(10 pts) (a) It is impossible to find c_1, c_2, c_3 such that $f(t)$ fits all points in the dataset $\{(2, -7), (-2, 0), (0, 0), (-1, 0)\}$. Use the least squares technique to find $\widehat{f}(t) = \widehat{c}_1 \cdot f_1(t) + \widehat{c}_2 \cdot f_2(t) + \widehat{c}_3 \cdot f_3(t)$ that best fits this data. Record your values of $\widehat{c}_1$, $\widehat{c}_2$, and $\widehat{c}_3$ below for clarity.

$\widehat{c}_1 =$ ________, $\widehat{c}_2 =$ ________, $\widehat{c}_3 =$ ________

(8 pts) (b) It is also impossible to find c_1, c_2, and c_3 such that $f(t)$ fits all points in the dataset $\{(0, -2), (1, 0), (2, 1), (3, 2)\}$. The least squares version of $f(t)$ that best fits this dataset is $\widehat{f}(t) = f_1(t) - 2\, f_2(t) + f_3(t)$. Find the error E in this approximation. Record your value of E below for clarity.

$E =$ ________