Duke University

Math 218D-2

Matrices and Vectors

# Exam I

*Name:* Solutions

*Unique ID:* ______

*I have adhered to the Duke Community Standard in completing this exam.*

Signature: ______

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- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

**Problem 1.** The equation below depicts the result of multiplying two matrices $A$ and $B$.

$$\overset{A}{\begin{bmatrix} \text{——} & \boldsymbol{r}_1^\intercal & \text{——} \\ \text{——} & \boldsymbol{v}^\intercal & \text{——} \\ \text{——} & \boldsymbol{r}_3^\intercal & \text{——} \\ \text{——} & \boldsymbol{r}_4^\intercal & \text{——} \\ \text{——} & \boldsymbol{r}_5^\intercal & \text{——} \end{bmatrix}} \overset{B}{\begin{bmatrix} | & | & | & | \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{v} & \boldsymbol{b}_4 \\ | & | & | & | \end{bmatrix}} = \begin{bmatrix} -5 & 16 & -4 & 6 \\ 3 & -3 & 7 & -1 \\ -9 & 9 & -7 & 0 \\ -1 & -3 & 10 & 4 \\ 6 & 0 & 1 & 5 \end{bmatrix}$$

Note that the rows of $A$ are labeled as $\boldsymbol{r}_1, \boldsymbol{v}, \boldsymbol{r}_3, \boldsymbol{r}_4, \boldsymbol{r}_5$ and that the columns of $B$ are marked as $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{v}, \boldsymbol{b}_4$. In particular, note that the second row of $A$ is equal to the third column of $B$ and this vector is labeled as $\boldsymbol{v}$.

(6 pts) (*a*) $\langle \boldsymbol{r}_1, \boldsymbol{b}_1 \rangle =$ <u>$-5$</u>, $\langle \boldsymbol{r}_4, \boldsymbol{b}_2 \rangle =$ <u>$-3$</u>, and $\boldsymbol{r}_3^\intercal \boldsymbol{b}_4 =$ <u>$0$</u>

(2 pts) (*b*) Which, if any, of the following vectors is *orthogonal* to $\boldsymbol{r}_5$? Select all the apply (no partial credit).

○ $\boldsymbol{b}_1$ √ $\boldsymbol{b}_2$ ○ $\boldsymbol{v}$ ○ $\boldsymbol{b}_4$ ○ none of these

(2 pts) (*c*) Which, if any, of the following vectors forms an *obtuse* angle with $\boldsymbol{r}_5$? Select all the apply (no partial credit).

○ $\boldsymbol{b}_1$ ○ $\boldsymbol{b}_2$ ○ $\boldsymbol{v}$ ○ $\boldsymbol{b}_4$ √ none of these

(4 pts) (*d*) Let $\theta$ be the angle between $\boldsymbol{v}$ and $\boldsymbol{b}_4$. Then $\|\boldsymbol{v}\| \cdot \|\boldsymbol{b}_4\| \cdot \cos(\theta) =$ <u>$-1$</u>.

(6 pts) (*e*) $\|\boldsymbol{v}\| =$ <u>$\sqrt{7}$</u> and the $(3, 3)$ entry of $B^\intercal B$ is <u>$7$</u>

(3 pts) (*f*) Calculate this matrix product: $\begin{bmatrix} \text{——} & \boldsymbol{r}_5^\intercal & \text{——} \\ \text{——} & \boldsymbol{r}_3^\intercal & \text{——} \end{bmatrix} \begin{bmatrix} | & | \\ \boldsymbol{b}_4 & \boldsymbol{b}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 0 & -9 \end{bmatrix}$

(6 pts) (*g*) Calculate the matrix-vector product $A(\boldsymbol{b}_1 + \boldsymbol{b}_4)$. Clearly explain your reasoning to receive credit. Fill in the blank vector at the bottom of this page to make your answer clear.

**Solution.** The vector $\boldsymbol{b}_1 + \boldsymbol{b}_4$ is the same as the matrix-vector product

$$\boldsymbol{b}_1 + \boldsymbol{b}_4 = \overset{B}{\begin{bmatrix} | & | & | & | \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{v} & \boldsymbol{b}_4 \\ | & | & | & | \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Our desired matrix-vector product is then

$$A(\boldsymbol{b}_1 + \boldsymbol{b}_4) = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} & B & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 & 16 & -4 & 6 \\ 3 & -3 & 7 & -1 \\ -9 & 9 & -7 & 0 \\ -1 & -3 & 10 & 4 \\ 6 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -9 \\ 3 \\ 11 \end{bmatrix}$$

The next problem references the concept of a *skew-symmetric matrix*, which is any matrix $A$ satisfying $A^\intercal = -A$.

**Problem 2.** Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two vectors in $\mathbb{R}^n$, both viewed as $n \times 1$ matrices. Let $K$ be the matrix defined by the formula $K = \boldsymbol{v}\boldsymbol{w}^\intercal - \boldsymbol{w}\boldsymbol{v}^\intercal$.

(10 pts) (*a*) Show that $K$ is skew-symmetric. Your solution should consist of a single string of equalities that is clear and coherent and avoids circular reasoning.

**Solution.** We are given that $K = \boldsymbol{v}\boldsymbol{w}^\intercal - \boldsymbol{w}\boldsymbol{v}^\intercal$. We wish to demonstrate that $K$ is skew-symmetric, which means that we want to show that $K^\intercal = -K$. To do so, note that

$$K^\intercal = (\boldsymbol{v}\boldsymbol{w}^\intercal - \boldsymbol{w}\boldsymbol{v}^\intercal)^\intercal = (\boldsymbol{v}\boldsymbol{w}^\intercal)^\intercal - (\boldsymbol{w}\boldsymbol{v}^\intercal)^\intercal = (\boldsymbol{w}^\intercal)^\intercal\boldsymbol{v}^\intercal - (\boldsymbol{v}^\intercal)^\intercal\boldsymbol{w}^\intercal = \boldsymbol{w}\boldsymbol{v}^\intercal - \boldsymbol{v}\boldsymbol{w}^\intercal = -(\boldsymbol{v}\boldsymbol{w}^\intercal - \boldsymbol{w}\boldsymbol{v}^\intercal) = -K$$

(10 pts) (*b*) Now, assume that $\|v\| = 3$, $\|w\| = 5$, and that $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal. Calculate $\|K\boldsymbol{v}\|$. *Hint.* Start by simplifying $K\boldsymbol{v}$ as far as possible before calculating the length of this vector.

**Solution.** The important fact from class is that $\boldsymbol{x}^\intercal\boldsymbol{y} = \langle\boldsymbol{x}, \boldsymbol{y}\rangle$. Here, we have

$$K\boldsymbol{v} = (\boldsymbol{v}\boldsymbol{w}^\intercal - \boldsymbol{w}\boldsymbol{v}^\intercal)\boldsymbol{v} = \boldsymbol{v}\boldsymbol{w}^\intercal\boldsymbol{v} - \boldsymbol{w}\boldsymbol{v}^\intercal\boldsymbol{v} = \boldsymbol{v}\langle\boldsymbol{w}, \boldsymbol{v}\rangle - \boldsymbol{w}\langle\boldsymbol{v}, \boldsymbol{v}\rangle = -\|\boldsymbol{v}\|^2\boldsymbol{w} = -9 \cdot \boldsymbol{w}$$

It follows that $\|K\boldsymbol{v}\| = \|-9 \cdot \boldsymbol{w}\| = 9\|\boldsymbol{w}\| = 9 \cdot 5 = 45$.

(12 pts) **Problem 3.** Fill-in the blank next to each of the following matrices with the appropriate notation to indicate the first step called for by the Gauß-Jordan algorithm as articulated in class. You do not need to perform the calculation but you must use correct notation to receive credit. (No partial credit. 2pts each)

$$\begin{bmatrix} 1 & -7 & 5 & 17 & 4 \\ 0 & 0 & 0 & 6 & 10 \\ 0 & 0 & -101 & 5 & 9 \\ 0 & 0 & 13 & 2 & 5 \\ 0 & 0 & 1 & 0 & 5 \end{bmatrix} \quad \underline{\boldsymbol{r}_2 \leftrightarrow \boldsymbol{r}_3}$$

$$\begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \underline{\boldsymbol{r}_1 - 7 \cdot \boldsymbol{r}_2 \to \boldsymbol{r}_1}$$

$$\begin{bmatrix} 1 & 0 & -3 & 7 & 13 \\ 0 & 1 & 9 & 6 & 4 \\ 0 & 0 & 0 & 17 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \quad \underline{(1/17) \cdot \boldsymbol{r}_3 \to \boldsymbol{r}_3}$$

$$\begin{bmatrix} 0 & 3 & 7 & 19 & 4 \\ 0 & 1 & 7 & 5 & 3 \\ 0 & 0 & 0 & 4 & 9 \\ 0 & -8 & 5 & 4 & 2 \end{bmatrix} \quad \underline{\tfrac{1}{3} \cdot \boldsymbol{r}_1 \to \boldsymbol{r}_1}$$

$$\begin{bmatrix} 156 & 361 & 373 \\ 691 & 383 & 172 \\ 443 & 681 & 541 \\ 1 & 866 & 776 \\ 229 & 944 & 892 \end{bmatrix} \quad \underline{(1/156) \cdot \boldsymbol{r}_1 \to \boldsymbol{r}_1}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \underline{\boldsymbol{r}_1 \leftrightarrow \boldsymbol{r}_4}$$

**Problem 4.** The following system of linear equations is consistent and in row echelon form.

$$\begin{array}{rcrcrcrcrcr} -2\,x_1 & + & 6\,x_2 & + & 3\,x_3 & - & 4\,x_4 & + & 5\,x_5 & = & -35 \\ & & & & 4\,x_3 & + & 8\,x_4 & - & 4\,x_5 & = & 4 \\ & & & & & & 6\,x_4 & + & 18\,x_5 & = & 24 \end{array}$$

Let $A$ be the coefficient matrix of this system.

(2 pts) (*a*) Which variables in this system is *dependent*? Select all that apply (no partial credit).

√ $x_1$ ○ $x_2$ √ $x_3$ √ $x_4$ ○ $x_5$

(6 pts) (*b*) Which of the following statements is true? Select all that apply (1.5pts each).

√ $A$ is full row rank ○ $A$ is full column rank ○ $A$ is rank deficient ○ $A^\intercal A$ is invertible

(10 pts) (*c*) Use back-substitution to express the general solution to this system as $\boldsymbol{x} = \boldsymbol{x}_p + c_1 \cdot \boldsymbol{x}_1 + c_2 \cdot \boldsymbol{x}_2$. Clearly explain your reasoning to receive credit. Fill in the blank vectors at the bottom of this page to make your answer clear.

**Solution.** The dependent variables are $x_1, x_3, x_4$ and the free variables are

$$x_2 = c_1 \qquad\qquad x_5 = c_2$$

We start by using the last equation to solve for $x_4$ with

$$x_4 = \frac{24 - 18\,c_2}{6} = 4 - 3\,c_2$$

We then use this formula for $x_4$ in the second equation and solve for $x_3$ with

$$\begin{aligned} x_3 &= \frac{4 - 8\,(4 - 3\,c_2) + 4\,c_2}{4} \\ &= 1 - 2\,(4 - 3\,c_2) + c_2 \\ &= 1 - 8 + 6\,c_2 + c_2 \\ &= -7 + 7\,c_2 \end{aligned}$$

Finally, we use our formulas for $x_3$ and $x_4$ in the first equation to solve for $x_1$ with

$$\begin{aligned} x_1 &= \frac{-35 - 6\,c_1 - 3\,(-7 + 7\,c_2) + 4\,(4 - 3\,c_2) - 5\,c_2}{-2} \\ &= \frac{-35 - 6\,c_1 + 21 - 21\,c_2 + 16 - 12\,c_2 - 5\,c_2}{-2} \\ &= \frac{(-35 + 21 + 16) - 6\,c_1 + (-21 - 12 - 5)\,c_2}{-2} \\ &= \frac{2 - 6\,c_1 - 38\,c_2}{-2} \\ &= -1 + 3\,c_1 + 19\,c_2 \end{aligned}$$

Putting all this together gives our general solution

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 + 3\,c_1 + 19\,c_2 \\ c_1 \\ -7 + 7\,c_2 \\ 4 - 3\,c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -7 \\ 4 \\ 0 \end{bmatrix} + c_1 \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 19 \\ 0 \\ 7 \\ -3 \\ 1 \end{bmatrix}$$

**Problem 5.** The equation below depicts the product of a ***nonsingular*** $3 \times 3$ matrix $A$ with another matrix $B$.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\overbrace{\begin{bmatrix} 2 & 0 & -1 & 1 & 0 & 1 & 0 \\ -2 & -1 & 1 & -1 & 0 & 2 & -3 \\ 1 & -3 & 2 & 0 & 1 & -1 & -2 \end{bmatrix}}^{B} = \begin{bmatrix} 0 & 34 & -15 & 3 & -6 & -39 & 60 \\ 0 & -35 & 15 & -3 & 6 & 42 & -63 \\ 1 & -3 & 2 & 0 & 1 & -1 & -2 \end{bmatrix}$$

(4 pts) $(a)$ $\operatorname{rank}(A) =$ __3__, $\operatorname{nullity}(A) =$ __0__, and $\operatorname{rank}(A^\intercal A) =$ __3__

(3 pts) $(b)$ What is the last column of $A$?

○ $\begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}$ ○ $\begin{bmatrix} 60 \\ -63 \\ -2 \end{bmatrix}$ ✓ $\begin{bmatrix} -6 \\ 6 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ ○ $\begin{bmatrix} -15 \\ 15 \\ 2 \end{bmatrix}$

(3 pts) $(c)$ The vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda =$ __3__.

(3 pts) $(d)$ Which of the following vectors is equal to $\begin{bmatrix} & & \\ & A^{-1} & \\ & & \end{bmatrix}\begin{bmatrix} -39 \\ 42 \\ -1 \end{bmatrix}$?

✓ $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}$ ○ $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

(8 pts) $(e)$ Let $\boldsymbol{b} = \begin{bmatrix} -15 \\ 15 \\ 102 \end{bmatrix}$. Find the solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your reasoning to receive credit. Fill in the blank vector at the bottom of this page to make your answer clear. *Hint.* Note that $\boldsymbol{b} = \begin{bmatrix} -15 \\ 15 \\ 100 + 2 \end{bmatrix}$.

**Solution.** The solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\begin{aligned} \boldsymbol{x} &= \begin{bmatrix} & & \\ & A^{-1} & \\ & & \end{bmatrix}\overbrace{\begin{bmatrix} -15 \\ 15 \\ 102 \end{bmatrix}}^{\boldsymbol{b}} \\ &= \begin{bmatrix} & & \\ & A^{-1} & \\ & & \end{bmatrix}\left(100 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -15 \\ 15 \\ 2 \end{bmatrix}\right) \\ &= 100 \cdot \begin{bmatrix} & & \\ & A^{-1} & \\ & & \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} & & \\ & A^{-1} & \\ & & \end{bmatrix}\begin{bmatrix} -15 \\ 15 \\ 2 \end{bmatrix} \\ &= 100 \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 199 \\ -199 \\ 102 \end{bmatrix} \end{aligned}$$