Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: ______________________ *Unique ID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

February 7, 2025

- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke University MATH

Problem 1. The equation below depicts the result of multiplying two matrices A and B.

$$\overset{A}{\begin{bmatrix} \text{———} & \boldsymbol{r}_1^{\intercal} & \text{———} \\ \text{———} & \boldsymbol{v}^{\intercal} & \text{———} \\ \text{———} & \boldsymbol{r}_3^{\intercal} & \text{———} \\ \text{———} & \boldsymbol{r}_4^{\intercal} & \text{———} \\ \text{———} & \boldsymbol{r}_5^{\intercal} & \text{———} \end{bmatrix}} \overset{B}{\begin{bmatrix} | & | & | & | \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{v} & \boldsymbol{b}_4 \\ | & | & | & | \end{bmatrix}} = \begin{bmatrix} -5 & 16 & -4 & 6 \\ 3 & -3 & 7 & -1 \\ -9 & 9 & -7 & 0 \\ -1 & -3 & 10 & 4 \\ 6 & 0 & 1 & 5 \end{bmatrix}$$

Note that the rows of A are labeled as $\boldsymbol{r}_1, \boldsymbol{v}, \boldsymbol{r}_3, \boldsymbol{r}_4, \boldsymbol{r}_5$ and that the columns of B are marked as $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{v}, \boldsymbol{b}_4$. In particular, note that the second row of A is equal to the third column of B and this vector is labeled as $\boldsymbol{v}$.

(6 pts) (*a*) $\langle \boldsymbol{r}_1, \boldsymbol{b}_1 \rangle =$ _______, $\langle \boldsymbol{r}_4, \boldsymbol{b}_2 \rangle =$ _______, and $\boldsymbol{r}_3^{\intercal}\boldsymbol{b}_4 =$ _______

(2 pts) (*b*) Which, if any, of the following vectors is *orthogonal* to $\boldsymbol{r}_5$? Select all the apply (no partial credit).

○ $\boldsymbol{b}_1$ ○ $\boldsymbol{b}_2$ ○ $\boldsymbol{v}$ ○ $\boldsymbol{b}_4$ ○ none of these

(2 pts) (*c*) Which, if any, of the following vectors forms an *obtuse* angle with $\boldsymbol{r}_5$? Select all the apply (no partial credit).

○ $\boldsymbol{b}_1$ ○ $\boldsymbol{b}_2$ ○ $\boldsymbol{v}$ ○ $\boldsymbol{b}_4$ ○ none of these

(4 pts) (*d*) Let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{b}_4$. Then $\|\boldsymbol{v}\| \cdot \|\boldsymbol{b}_4\| \cdot \cos(\theta) =$ _______.

(6 pts) (*e*) $\|\boldsymbol{v}\| =$ _______ and the $(3, 3)$ entry of $B^{\intercal}B$ is _______

(3 pts) (*f*) Calculate this matrix product: $\begin{bmatrix} \text{———} & \boldsymbol{r}_5^{\intercal} & \text{———} \\ \text{———} & \boldsymbol{r}_3^{\intercal} & \text{———} \end{bmatrix} \begin{bmatrix} | & | \\ \boldsymbol{b}_4 & \boldsymbol{b}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

(6 pts) (*g*) Calculate the matrix-vector product $A(\boldsymbol{b}_1 + \boldsymbol{b}_4)$. Clearly explain your reasoning to receive credit. Fill in the blank vector at the bottom of this page to make your answer clear.

$$A(\boldsymbol{b}_1 + \boldsymbol{b}_4) = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

The next problem references the concept of a *skew-symmetric matrix*, which is any matrix A satifying $A^\intercal = -A$.

Problem 2. Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two vectors in $\mathbb{R}^n$, both viewed as $n \times 1$ matrices. Let K be the matrix defined by the formula $K = \boldsymbol{v}\boldsymbol{w}^\intercal - \boldsymbol{w}\boldsymbol{v}^\intercal$.

(10 pts) (*a*) Show that K is skew-symmetric. Your solution should consist of a single string of equalities that is clear and coherent and avoids circular reasoning.

(10 pts) (*b*) Now, assume that $\|v\| = 3$, $\|w\| = 5$, and that $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal. Calculate $\|K\boldsymbol{v}\|$. *Hint.* Start by simplifying $K\boldsymbol{v}$ as far as possible before calculating the length of this vector.

(12 pts) **Problem 3.** Fill-in the blank next to each of the following matrices with the appropriate notation to indicate the first step called for by the Gauß-Jordan algorithm as articulated in class. You do not need to perform the calculation but you must use correct notation to receive credit. (No partial credit. 2pts each)

$$\begin{bmatrix} 1 & -7 & 5 & 17 & 4 \\ 0 & 0 & 0 & 6 & 10 \\ 0 & 0 & -101 & 5 & 9 \\ 0 & 0 & 13 & 2 & 5 \\ 0 & 0 & 1 & 0 & 5 \end{bmatrix} \underline{\hspace{4cm}} \qquad \begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{\hspace{4cm}}$$

$$\begin{bmatrix} 1 & 0 & -3 & 7 & 13 \\ 0 & 1 & 9 & 6 & 4 \\ 0 & 0 & 0 & 17 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \underline{\hspace{4cm}} \qquad \begin{bmatrix} 0 & 3 & 7 & 19 & 4 \\ 0 & 1 & 7 & 5 & 3 \\ 0 & 0 & 0 & 4 & 9 \\ 0 & -8 & 5 & 4 & 2 \end{bmatrix} \underline{\hspace{4cm}}$$

$$\begin{bmatrix} 156 & 361 & 373 \\ 691 & 383 & 172 \\ 443 & 681 & 541 \\ 1 & 866 & 776 \\ 229 & 944 & 892 \end{bmatrix} \underline{\hspace{4cm}} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{\hspace{4cm}}$$

Problem 4. The following system of linear equations is consistent and in row echelon form.

$$\begin{array}{rcrcrcrcrcr} -2\,x_1 & + & 6\,x_2 & + & 3\,x_3 & - & 4\,x_4 & + & 5\,x_5 & = & -35 \\ & & & & 4\,x_3 & + & 8\,x_4 & - & 4\,x_5 & = & 4 \\ & & & & & & 6\,x_4 & + & 18\,x_5 & = & 24 \end{array}$$

Let A be the coefficient matrix of this system.

(2 pts) (*a*) Which variables in this system is *dependent*? Select all that apply (no partial credit).

◯ x_1 ◯ x_2 ◯ x_3 ◯ x_4 ◯ x_5

(6 pts) (*b*) Which of the following statements is true? Select all that apply (1.5pts each).

◯ A is full row rank ◯ A is full column rank ◯ A is rank deficient ◯ $A^\intercal A$ is invertible

(10 pts) (*c*) Use back-substitution to express the general solution to this system as $\boldsymbol{x} = \boldsymbol{x}_p + c_1 \cdot \boldsymbol{x}_1 + c_2 \cdot \boldsymbol{x}_2$. Clearly explain your reasoning to receive credit. Fill in the blank vectors at the bottom of this page to make your answer clear.

$$\boldsymbol{x} = \overset{\boldsymbol{x}_p}{\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}} + c_1 \cdot \overset{\boldsymbol{x}_1}{\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}} + c_2 \cdot \overset{\boldsymbol{x}_2}{\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}}$$

Problem 5. The equation below depicts the product of a ***nonsingular*** 3×3 matrix A with another matrix B.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\overbrace{\begin{bmatrix} 2 & 0 & -1 & 1 & 0 & 1 & 0 \\ -2 & -1 & 1 & -1 & 0 & 2 & -3 \\ 1 & -3 & 2 & 0 & 1 & -1 & -2 \end{bmatrix}}^{B} = \begin{bmatrix} 0 & 34 & -15 & 3 & -6 & -39 & 60 \\ 0 & -35 & 15 & -3 & 6 & 42 & -63 \\ 1 & -3 & 2 & 0 & 1 & -1 & -2 \end{bmatrix}$$

(4 pts) (*a*) $\operatorname{rank}(A) =$ _______, $\operatorname{nullity}(A) =$ _______, and $\operatorname{rank}(A^\intercal A) =$ _______

(3 pts) (*b*) What is the last column of A?

○ $\begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}$ ○ $\begin{bmatrix} 60 \\ -63 \\ -2 \end{bmatrix}$ ○ $\begin{bmatrix} -6 \\ 6 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ ○ $\begin{bmatrix} -15 \\ 15 \\ 2 \end{bmatrix}$

(3 pts) (*c*) The vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda =$ _______.

(3 pts) (*d*) Which of the following vectors is equal to $\begin{bmatrix} & & \\ & A^{-1} & \\ & & \end{bmatrix}\begin{bmatrix} -39 \\ 42 \\ -1 \end{bmatrix}$?

○ $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}$ ○ $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

(8 pts) (*e*) Let $\boldsymbol{b} = \begin{bmatrix} -15 \\ 15 \\ 102 \end{bmatrix}$. Find the solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your reasoning to receive credit. Fill in the blank vector at the bottom of this page to make your answer clear. *Hint.* Note that $\boldsymbol{b} = \begin{bmatrix} -15 \\ 15 \\ 100+2 \end{bmatrix}$.

$$\boldsymbol{x} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$