Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: Solutions

Unique ID: ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________________________

March 7, 2025

- There are 100 points and 6 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider $\overset{P}{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}}\overset{A}{\begin{bmatrix}2&5&1&2\\-4&-10&-2&-5\\10&25&8&10\end{bmatrix}}=\overset{L}{\begin{bmatrix}1&0&0\\5&1&0\\-2&0&1\end{bmatrix}}\overset{U}{\begin{bmatrix}2&5&1&2\\0&0&3&0\\0&0&0&-1\end{bmatrix}}$.

(10 pts) (*a*) Use the algorithm discussed in class to calculate the matrices P, L, and U. Fill in the blank matrices above to make your answer clear. ***To receive points your work must be neatly organized and easy to follow.***

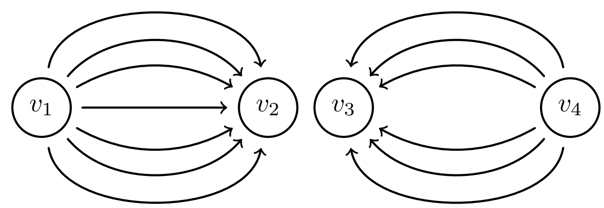
Solution. Following the algorithm from class, we have

$$\overset{A}{\begin{bmatrix}2&5&1&2\\-4&-10&-2&-5\\10&25&8&10\end{bmatrix}}\xrightarrow{\substack{\boldsymbol{r}_2\ +\ 2\cdot\boldsymbol{r}_1\ \to\ \boldsymbol{r}_2\\ \boldsymbol{r}_3\ -\ 5\cdot\boldsymbol{r}_1\ \to\ \boldsymbol{r}_3}}\overset{U}{\begin{bmatrix}2&5&1&2\\0&0&0&-1\\0&0&3&0\end{bmatrix}}\quad\overset{L}{\begin{bmatrix}0&0&0\\-2&0&0\\5&0&0\end{bmatrix}}\quad\overset{P}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}$$

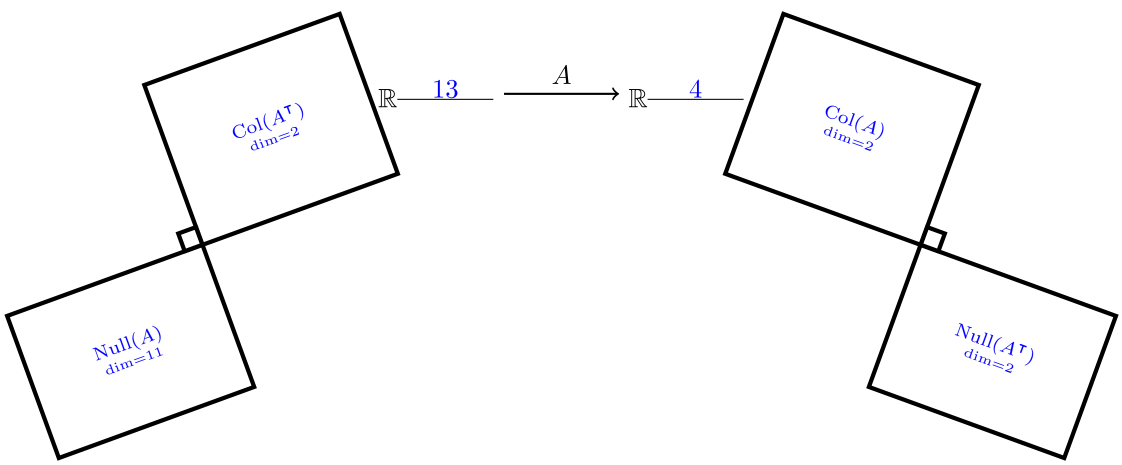
$$\xrightarrow{\boldsymbol{r}_2\leftrightarrow\boldsymbol{r}_3}\begin{bmatrix}2&5&1&2\\0&0&3&0\\0&0&0&-1\end{bmatrix}\quad\begin{bmatrix}0&0&0\\5&0&0\\-2&0&0\end{bmatrix}\quad\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

That's it! To finish we put 1's on the diagonal of L.

Problem 2. Let A be the incidence matrix of this directed graph



(10 pts) (*a*) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(4 pts) (*b*) Let x and y be scalars. Only one of the following vectors is *guaranteed* to be orthogonal to the column space of A. Select this vector. ○ $\begin{bmatrix}x\\-x\\y\\-y\end{bmatrix}$ ○ $\begin{bmatrix}x\\y\\x\\y\end{bmatrix}$ ○ $\begin{bmatrix}x\\y\\-x\\-y\end{bmatrix}$ ○ $\begin{bmatrix}x\\-y\\x\\-y\end{bmatrix}$ ✓ $\begin{bmatrix}x\\x\\y\\y\end{bmatrix}$

Problem 3. Suppose $\boldsymbol{v}_1 + 3 \cdot \boldsymbol{v}_2 + 0 \cdot \boldsymbol{v}_3 - 5 \cdot \boldsymbol{v}_4 = \boldsymbol{w}$ where $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 \in \mathbb{R}^5$ are linearly independent.

(6 pts) (*a*) $\operatorname{rank}\begin{bmatrix} | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \\ | & | & | & | \end{bmatrix} = \underline{\quad 4 \quad}$, $\operatorname{rank}\begin{bmatrix} | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 \\ | & | \end{bmatrix} = \underline{\quad 2 \quad}$, and $\operatorname{rank}\begin{bmatrix} | & | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 & \boldsymbol{w} \\ | & | & | & | & | \end{bmatrix} = \underline{\quad 4 \quad}$

(4 pts) (*b*) Only one of the following vectors is in the null space of $\begin{bmatrix} | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_4 & \boldsymbol{w} \\ | & | & | & | \end{bmatrix}$. Select this vector.

○ $\begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \\ 0 \\ -5 \\ -1 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ -3 \\ 5 \\ -1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \\ 0 \\ -5 \end{bmatrix}$ ✓ $\begin{bmatrix} 1 \\ 3 \\ -5 \\ -1 \end{bmatrix}$

(4 pts) (*c*) Only one of the following vectors is in the *left* null space of $\begin{bmatrix} \text{---} & \boldsymbol{w}^\intercal & \text{---} \\ \text{---} & \boldsymbol{v}_2^\intercal & \text{---} \\ \text{---} & \boldsymbol{v}_1^\intercal & \text{---} \\ \text{---} & \boldsymbol{v}_4^\intercal & \text{---} \end{bmatrix}$. Select this vector.

○ $\begin{bmatrix} 1 \\ -1 \\ -3 \\ 5 \end{bmatrix}$ ✓ $\begin{bmatrix} 1 \\ -3 \\ -1 \\ 5 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \\ 0 \\ -5 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 5 \\ -3 \\ -1 \end{bmatrix}$

(6 pts) (*d*) $\operatorname{rref}\begin{bmatrix} | & | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_4 & \boldsymbol{w} & \boldsymbol{v}_3 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (note that this matrix is 5×5).

Solution. The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are independent and we know that $\boldsymbol{w} = \boldsymbol{v}_1 + 3\boldsymbol{v}_2 - \boldsymbol{v}_4$. This means the first, second, third, and fifth columns of this matrix are pivot columns and the fourth is a nonpivot column conforming to the column relation $\text{Col}_4 = \text{Col}_1 + 3\ \text{Col}_2 - 5\ \text{Col}_3$.

Problem 4. Let P be the projection matrix onto a vector space $V \subset \mathbb{R}^n$, let $\boldsymbol{v}$ be any vector in $\mathbb{R}^n$, and let θ be the angle between $\boldsymbol{v}$ and $P\boldsymbol{v}$.

(10 pts) (*a*) Show that $\|P\boldsymbol{v}\|^2 = \|\boldsymbol{v}\| \cdot \|P\boldsymbol{v}\| \cdot \cos(\theta)$. Your solution should consist of a single string of equalities that is clear and coherent and avoids circular reasoning.

Solution. We know that P is *symmetric* ($P^\intercal = P$) and *idempotent* ($P^2 = P$). It follows that

$$\|P\boldsymbol{v}\|^2 = \langle P\boldsymbol{v}, P\boldsymbol{v}\rangle = \langle \boldsymbol{v}, P^\intercal P\boldsymbol{v}\rangle = \langle \boldsymbol{v}, P^2\boldsymbol{v}\rangle = \langle \boldsymbol{v}, P\boldsymbol{v}\rangle = \|\boldsymbol{v}\| \cdot \|P\boldsymbol{v}\| \cdot \cos(\theta)$$

Here, the second equality is justified by the adjoint property of inner products and the penultimate equality is the geometric formula for inner products.

(4 pts) (*b*) The fact that $\|P\boldsymbol{v}\|^2 = \|\boldsymbol{v}\| \cdot \|P\boldsymbol{v}\| \cdot \cos(\theta)$ tells us that exactly one of the following statements about the angle θ between a vector $\boldsymbol{v}$ and its projection to any vector space $P\boldsymbol{v}$ is true. Select this fact.

○ θ must be acute ○ θ cannot be acute ○ θ must be obtuse ✓ θ cannot be obtuse ○ $\theta \neq \pi/2$

Problem 5. Each of the matrices in the $EA = R$ factorization below is a 5×5 matrix.

$$\overset{E}{\begin{bmatrix} -2 & -1 & 3 & -1 & -1 \\ 4 & 0 & -4 & 2 & -1 \\ -2 & 0 & 2 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 & -1 & 4 & 5 \\ 0 & 1 & 2 & -6 & -6 \\ 1 & 0 & -1 & 4 & 5 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & -2 & 6 & 6 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & -4 & -4 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

It is known that $\text{E-Vals}(A) = \{0, 3, 5\}$. Throughout this problem, P will denote the projection matrix on to the *left null space* of A.

(7.5 pts) (*a*) Which of the following matrices *does not exist*? Select all that apply (1.5pts each).

○ E^{-1} √ A^{-1} √ R^{-1} √ $(3 \cdot I_5 - A)^{-1}$ ○ $(3 \cdot I_5 - R)^{-1}$

(3.5 pts) (*b*) Only one of the following statements accurately describes the relationship between the eigenvalues of A and the eigenvalues of R. Select this statement.

○ A and R have no common eigenvalues √ A and R share exactly one eigenvalue

○ A and R have exactly the same eigenvalues, but their geometric multiplicities are different

○ A and R have exactly the same eigenvalues with exactly the same geometric multiplicities

○ A and R share exactly two eigenvalues

(4 pts) (*c*) Only one of the following vectors is *guaranteed* to be in the column space of A for every scalar value x. Select this vector.

○ $\begin{bmatrix} x \\ x \\ 0 \\ 0 \\ -x \end{bmatrix}$ ○ $\begin{bmatrix} x \\ x \\ x \\ x \\ x \end{bmatrix}$ ○ $\begin{bmatrix} x \\ x \\ x \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} x \\ x \\ -x \\ 0 \\ x \end{bmatrix}$ √ $\begin{bmatrix} x \\ x \\ x \\ x \\ -x \end{bmatrix}$

(5 pts) (*d*) $\operatorname{trace}(P) =$ __2__ and $\dim \mathcal{E}_A(0) =$ __2__

(8 pts) (*e*) Calculate the matrix P and fill in the blank matrix at the bottom of this page to make your answer clear. You must clearly explain your reasoning to receive credit.

Solution. We know that $P = X(X^\intercal X)^{-1}X^\intercal$ where X is any matrix whose columns form a basis of the left null space of A. Note that there are two rows of zeros at the bottom of R. We learned in class that the bottom two rows of E then form a basis of $\operatorname{Null}(A^\intercal)$ so an appropriate choice of X is

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad X^\intercal X = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2 \cdot I_2 \qquad (X^\intercal X)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = (\frac{1}{2})I_2$$

We have calculated $X^\intercal X = 2 \cdot I_2$ and therefore $(X^\intercal X)^{-1} = (1/2) \cdot I_2$. Our desired projection matrix is then

$$P = \overset{X}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}} \overset{(X^\intercal X)^{-1}}{(1/2) \cdot I_2} \overset{X^\intercal}{\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Problem 6. Let A be the matrix

$$A = \begin{bmatrix} -10 & 9 & -2 & 9 & 4 \\ -22 & 19 & -4 & 18 & 8 \\ 6 & -6 & 1 & -6 & -3 \\ 4 & -4 & 0 & -3 & -2 \\ 16 & -12 & 4 & -12 & -4 \end{bmatrix}$$

It is known that $\lambda = 1$ is an eigenvalue of A.

(10 pts) (a) Find two linearly independent vectors in $\mathcal{E}_A(1)$. Clearly explain your reasoning to receive credit. Fill in the blank vectors at the bottom of this page to make your answer clear.

Solution. We want two linearly independent vectors in $\mathcal{E}_A(1) = \text{Null}(I - A)$, which is

$$\mathcal{E}_A(1) = \text{Null} \overbrace{\begin{bmatrix} 11 & -9 & 2 & -9 & -4 \\ 22 & -18 & 4 & -18 & -8 \\ -6 & 6 & 0 & 6 & 3 \\ -4 & 4 & 0 & 4 & 2 \\ -16 & 12 & -4 & 12 & 5 \end{bmatrix}}^{I-A}$$

The first two columns of $I - A$ are not multiples of each other and are thus independent. The third column is the sum of the first two and the fourth equals the second. This gives two column relations

$$\text{Col}_3 = \text{Col}_1 + \text{Col}_2 \qquad \text{Col}_4 = \text{Col}_2$$

These column relations provide us with two vectors in $\mathcal{E}_A(1) = \text{Null}(I - A)$ given by

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

These vectors are independent because they are not multiples of each other. We have thus successfully found two independent vectors in $\mathcal{E}_A(1)$.

(4 pts) (b) The fact that it is possible to find two linearly independent eigenvectors in $\mathcal{E}_A(1)$ tells us that exactly one of the following statements is true. Select this statement.

√ $\text{gm}_A(1) \geq 2$ ○ $\text{gm}_A(1) = 2$ ○ $\text{gm}_A(1) < 2$ ○ none of these