Duke University

Math 218D-2

Matrices and Vectors

# Exam II

*Name:* ____________________ *Unique ID:* ____________________

*I have adhered to the Duke Community Standard in completing this exam.*

Signature: ____________________

March 7, 2025

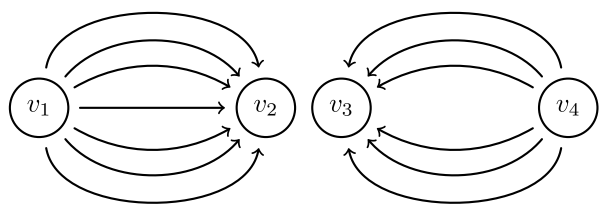
- There are 100 points and 6 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

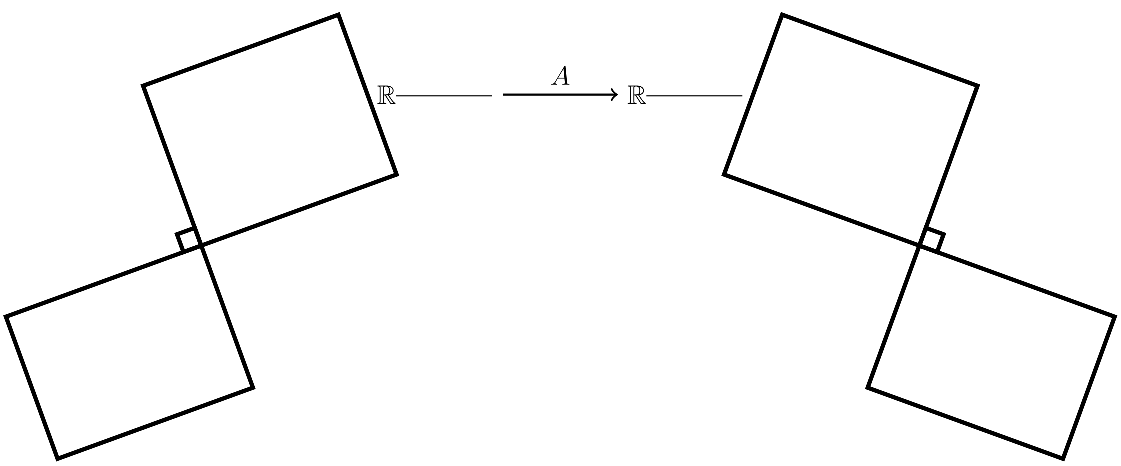
**Problem 1.** Consider $\overset{P}{\begin{bmatrix} \phantom{0} & \phantom{0} & \phantom{0} \\ & & \\ & & \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 5 & 1 & 2 \\ -4 & -10 & -2 & -5 \\ 10 & 25 & 8 & 10 \end{bmatrix}} = \overset{L}{\begin{bmatrix} \phantom{0} & \phantom{0} & \phantom{0} \\ & & \\ & & \end{bmatrix}} \overset{U}{\begin{bmatrix} \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \\ & & & \\ & & & \end{bmatrix}}$.

(10 pts) ($a$) Use the algorithm discussed in class to calculate the matrices $P$, $L$, and $U$. Fill in the blank matrices above to make your answer clear. ***To receive points your work must be neatly organized and easy to follow.***

**Problem 2.** Let $A$ be the incidence matrix of this directed graph




(10 pts) ($a$) Fill in every missing label in the picture of the four fundamental subspaces of $A$ below, including the dimension of each fundamental subspace.




(4 pts) ($b$) Let $x$ and $y$ be scalars. Only one of the following vectors is *guaranteed* to be orthogonal to the column space of $A$. Select this vector. ○ $\begin{bmatrix} x \\ -x \\ y \\ -y \end{bmatrix}$ ○ $\begin{bmatrix} x \\ y \\ x \\ y \end{bmatrix}$ ○ $\begin{bmatrix} x \\ y \\ -x \\ -y \end{bmatrix}$ ○ $\begin{bmatrix} x \\ -y \\ x \\ -y \end{bmatrix}$ ○ $\begin{bmatrix} x \\ x \\ y \\ y \end{bmatrix}$

**Problem 3.** Suppose $\boldsymbol{v}_1 + 3 \cdot \boldsymbol{v}_2 + 0 \cdot \boldsymbol{v}_3 - 5 \cdot \boldsymbol{v}_4 = \boldsymbol{w}$ where $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 \in \mathbb{R}^5$ are linearly independent.

(6 pts) (*a*) $\operatorname{rank}\begin{bmatrix} | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \\ | & | & | & | \end{bmatrix} =$ _______, $\operatorname{rank}\begin{bmatrix} | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 \\ | & | \end{bmatrix} =$ _______, and $\operatorname{rank}\begin{bmatrix} | & | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 & \boldsymbol{w} \\ | & | & | & | & | \end{bmatrix} =$ _______

(4 pts) (*b*) Only one of the following vectors is in the null space of $\begin{bmatrix} | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_4 & \boldsymbol{w} \\ | & | & | & | \end{bmatrix}$. Select this vector.

○ $\begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \\ 0 \\ -5 \\ -1 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ -3 \\ 5 \\ -1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \\ 0 \\ -5 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \\ -5 \\ -1 \end{bmatrix}$

(4 pts) (*c*) Only one of the following vectors is in the *left* null space of $\begin{bmatrix} \text{---} & \boldsymbol{w}^\intercal & \text{---} \\ \text{---} & \boldsymbol{v}_2^\intercal & \text{---} \\ \text{---} & \boldsymbol{v}_1^\intercal & \text{---} \\ \text{---} & \boldsymbol{v}_4^\intercal & \text{---} \end{bmatrix}$. Select this vector.

○ $\begin{bmatrix} 1 \\ -1 \\ -3 \\ 5 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ -3 \\ -1 \\ 5 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \\ 0 \\ -5 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 5 \\ -3 \\ -1 \end{bmatrix}$

(6 pts) (*d*) $\operatorname{rref}\begin{bmatrix} | & | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_4 & \boldsymbol{w} & \boldsymbol{v}_3 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} \phantom{0} \end{bmatrix}$ (note that this matrix is $5 \times 5$).

**Problem 4.** Let $P$ be the projection matrix onto a vector space $V \subset \mathbb{R}^n$, let $\boldsymbol{v}$ be any vector in $\mathbb{R}^n$, and let $\theta$ be the angle between $\boldsymbol{v}$ and $P\boldsymbol{v}$.

(10 pts) (*a*) Show that $\|P\boldsymbol{v}\|^2 = \|\boldsymbol{v}\| \cdot \|P\boldsymbol{v}\| \cdot \cos(\theta)$. Your solution should consist of a single string of equalities that is clear and coherent and avoids circular reasoning.

(4 pts) (*b*) The fact that $\|P\boldsymbol{v}\|^2 = \|\boldsymbol{v}\| \cdot \|P\boldsymbol{v}\| \cdot \cos(\theta)$ tells us that exactly one of the following statements about the angle $\theta$ between a vector $\boldsymbol{v}$ and its projection to any vector space $P\boldsymbol{v}$ is true. Select this fact.

○ $\theta$ must be acute ○ $\theta$ cannot be acute ○ $\theta$ must be obtuse ○ $\theta$ cannot be obtuse ○ $\theta \neq {}^{\pi}/_{2}$

**Problem 5.** Each of the matrices in the $EA = R$ factorization below is a $5 \times 5$ matrix.

$$\overset{E}{\begin{bmatrix} -2 & -1 & 3 & -1 & -1 \\ 4 & 0 & -4 & 2 & -1 \\ -2 & 0 & 2 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 & -1 & 4 & 5 \\ 0 & 1 & 2 & -6 & -6 \\ 1 & 0 & -1 & 4 & 5 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & -2 & 6 & 6 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & -4 & -4 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

It is known that $\text{E-Vals}(A) = \{0, 3, 5\}$. Throughout this problem, $P$ will denote the projection matrix on to the *left null space* of $A$.

(7.5 pts) ($a$) Which of the following matrices *does not exist*? Select all that apply (1.5pts each).

○ $E^{-1}$ ○ $A^{-1}$ ○ $R^{-1}$ ○ $(3 \cdot I_5 - A)^{-1}$ ○ $(3 \cdot I_5 - R)^{-1}$

(3.5 pts) ($b$) Only one of the following statements accurately describes the relationship between the eigenvalues of $A$ and the eigenvalues of $R$. Select this statement.

○ $A$ and $R$ have no common eigenvalues ○ $A$ and $R$ share exactly one eigenvalue

○ $A$ and $R$ have exactly the same eigenvalues, but their geometric multiplicities are different

○ $A$ and $R$ have exactly the same eigenvalues with exactly the same geometric multiplicities

○ $A$ and $R$ share exactly two eigenvalues

(4 pts) ($c$) Only one of the following vectors is *guaranteed* to be in the column space of $A$ for every scalar value $x$. Select this vector.

○ $\begin{bmatrix} x \\ x \\ 0 \\ 0 \\ -x \end{bmatrix}$ ○ $\begin{bmatrix} x \\ x \\ x \\ x \\ x \end{bmatrix}$ ○ $\begin{bmatrix} x \\ x \\ x \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} x \\ x \\ -x \\ 0 \\ x \end{bmatrix}$ ○ $\begin{bmatrix} x \\ x \\ x \\ x \\ -x \end{bmatrix}$

(5 pts) ($d$) $\operatorname{trace}(P) =$ ______ and $\dim \mathcal{E}_A(0) =$ ______

(8 pts) ($e$) Calculate the matrix $P$ and fill in the blank matrix at the bottom of this page to make your answer clear. You must clearly explain your reasoning to receive credit.

$$P = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

**Problem 6.** Let $A$ be the matrix

$$A = \begin{bmatrix} -10 & 9 & -2 & 9 & 4 \\ -22 & 19 & -4 & 18 & 8 \\ 6 & -6 & 1 & -6 & -3 \\ 4 & -4 & 0 & -3 & -2 \\ 16 & -12 & 4 & -12 & -4 \end{bmatrix}$$

It is known that $\lambda = 1$ is an eigenvalue of $A$.

(10 pts) ($a$) Find two linearly independent vectors in $\mathcal{E}_A(1)$. Clearly explain your reasoning to receive credit. Fill in the blank vectors at the bottom of this page to make your answer clear.

$$\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} \text{ and } \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

(4 pts) ($b$) The fact that it is possible to find two linearly independent eigenvectors in $\mathcal{E}_A(1)$ tells us that exactly one of the following statements is true. Select this statement.

○ $\operatorname{gm}_A(1) \geq 2$ ○ $\operatorname{gm}_A(1) = 2$ ○ $\operatorname{gm}_A(1) < 2$ ○ none of these