DUKE UNIVERSITY

MATH 218D-2

MATRICES AND VECTORS

Exam III

Name: Solutions

Unique ID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

April 18, 2025

- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Suppose that $A = QR$ where A is 5×4 with $\operatorname{rank}(A) = 2$. Let $P = QQ^\intercal$ and let d be the degree of the characteristic polynomial of P.

(4 pts) (a) Q is __5__ $\times$ __2__ and R is __2__ $\times$ __4__

(5 pts) (b) $d =$ __5__ and the coefficient of t^{d-1} in $\chi_P(t)$ is __-2__

(4 pts) (c) $\operatorname{rank}(PA) =$ __2__ (note that $PA = QQ^\intercal QR = QIR = QR = A$)

Problem 2. Let $\boldsymbol{v} \in \mathbb{R}^2$ and let A be the matrix with eigenspaces $\mathcal{E}_A(-2) = \operatorname{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ and $\mathcal{E}_A(1) = \operatorname{Span}\{\boldsymbol{v}\}$.

(4 pts) (a) $\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-2\\-2\end{bmatrix}$

(4 pts) (b) Suppose $x \neq 0$. Only one of the following choices for the vector $\boldsymbol{v}$ makes the matrix A real-symmetric. Select this vector.

○ $\boldsymbol{v} = \begin{bmatrix}x\\x\end{bmatrix}$ √ $\boldsymbol{v} = \begin{bmatrix}x\\-x\end{bmatrix}$ ○ $\boldsymbol{v} = \begin{bmatrix}-x\\-x\end{bmatrix}$ ○ $\boldsymbol{v} = \begin{bmatrix}x\\0\end{bmatrix}$ ○ $\boldsymbol{v} = \begin{bmatrix}0\\x\end{bmatrix}$

(10 pts) (c) Calculate the matrix-vector product $\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$ assuming that $\boldsymbol{v} = \begin{bmatrix}2\\4\end{bmatrix}$.

Solution. The given data is exactly what we need to diagonalize A, which gives

$$\begin{aligned}
\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} &= \overset{X}{\begin{bmatrix}1 & 2\\1 & 4\end{bmatrix}}\overset{D}{\begin{bmatrix}-2 & 0\\0 & 1\end{bmatrix}}\overset{X^{-1}}{\frac{1}{2}\begin{bmatrix}4 & -2\\-1 & 1\end{bmatrix}}\begin{bmatrix}1\\3\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}1 & 2\\1 & 4\end{bmatrix}\begin{bmatrix}-2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-2\\2\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}1 & 2\\1 & 4\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}8\\12\end{bmatrix} \\
&= \begin{bmatrix}4\\6\end{bmatrix}
\end{aligned}$$

Problem 3. This 4×4 matrix A has exactly two eigenvalues. One of the eigenvalues of A is $\lambda_1 = 3$. The geometric and algebraic multiplicities of $\lambda_1 = 3$ satisfy

$$\operatorname{gm}_A(3) = \operatorname{am}_A(3) = 2$$

The other eigenvalue λ_2 is unknown. ***It is known that A is not diagonalizable.***

$$A = \begin{bmatrix} -3 & -41 & -4 & -6 \\ 3 & 24 & 2 & 3 \\ -12 & -86 & -5 & -12 \\ -9 & -63 & -6 & -6 \end{bmatrix}$$

(6 pts) (*a*) $\operatorname{gm}_A(\lambda_2) =$ __1__ and $\operatorname{am}_A(\lambda_2) =$ __2__

(10 pts) (*b*) Calculate the values of λ_2 and $\det(A)$. Clearly explain your reasoning to receive credit and *fill in the blanks below to make your answers clear.*

The context of this problem allows for the calculation of these numbers with minimal arithmetic, so no partial credit for arithmetic errors will be awarded.

Solution. The context of the problem tells us that the characteristic polynomial of A factors as

$$\chi_A(t) = (t-3)^2 \cdot (t-\lambda_2)^{\operatorname{am}_A(\lambda_2)}$$

Since A is 4×4, we must have $2 + \operatorname{am}_A(\lambda_2) = 4$, which means $\operatorname{am}_A(\lambda_2) = 2$ and

$$\chi_A(t) = (t-3)^2 \cdot (t-\lambda_2)^2$$

The trace property of eigenvalues then tells us that

$$10 = \operatorname{trace}(A) = 2 \cdot 3 + 2 \cdot \lambda_2$$

implying that $\lambda_2 = 2$. The determinant property of eigenvalues then tells us that $\det(A) = 3^2 \cdot 2^2 = 36$.

$\lambda_2 =$ __2__ and $\det(A) =$ __36__

(10 pts) (*c*) There is a basis of $\mathcal{E}_A(3)$ of the form $\{\boldsymbol{v}_1, \boldsymbol{v}_2 = \begin{bmatrix} -2 & 0 & 3 & 0 \end{bmatrix}^\intercal\}$. Find a valid choice for $\boldsymbol{v}_1$ and then use your basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to find an *orthonormal* basis $\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ of $\mathcal{E}_A(3)$. Clearly explain your reasoning to receive credit and *fill in the blank vectors below to make your answer clear.*

Solution. The eigenspace in question is

$$\mathcal{E}_A(3) = \operatorname{Null} \overbrace{\begin{bmatrix} 6 & 41 & 4 & 6 \\ -3 & -21 & -2 & -3 \\ 12 & 86 & 8 & 12 \\ 9 & 63 & 6 & 9 \end{bmatrix}}^{3 \cdot I_4 - A} = \operatorname{Span}\left\{ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$$

The first basis vector $\boldsymbol{v}_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^\intercal$ is easily found here by observing the column relation $\operatorname{Col}_1 = \operatorname{Col}_4$ in the characteristic matrix $3 \cdot I_4 - A$.

Now, of course, our orthonormal basis $\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ is found with Gram-Schmidt.

$$\begin{aligned} \boldsymbol{w}_1 &= \boldsymbol{v}_1 \\ \boldsymbol{w}_2 &= \boldsymbol{v}_2 - \operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_2) \\ &= \boldsymbol{v}_2 - \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_2 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 \\ &= \boldsymbol{v}_2 - \frac{-2}{2} \boldsymbol{w}_1 \\ &= \begin{bmatrix} -2 & 0 & 3 & 0 \end{bmatrix}^\intercal + \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^\intercal \\ &= \begin{bmatrix} -1 & 0 & 3 & -1 \end{bmatrix}^\intercal \end{aligned}$$

Normalizing then gives $\boldsymbol{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^\intercal$ and $\boldsymbol{q}_2 = \frac{1}{\sqrt{11}} \begin{bmatrix} -1 & 0 & 3 & -1 \end{bmatrix}^\intercal$.

Note that choosing $\boldsymbol{v}_1 = \begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}^\intercal$ is also valid and leads to the same $\boldsymbol{w}_2$ in the Gram-Schmidt process!

(16 pts) **Problem 4.** Let H be the Hermitian matrix $H = \begin{bmatrix} 2 & -i \\ i & 2 \end{bmatrix}$. Calculate $2 \cdot \exp(H)$. Clearly explain your reasoning and *fill in the blank matrix below to make you answer clear.*

Solution. This Hermitian matrix H admits a *spectral factorization* $H = UDU^*$. To find it, start with the characteristic polynomial of H, which is

$$\chi_H(t) = t^2 - \operatorname{trace}(H)\, t + \det(H) = t^2 - 4t + 3 = (t-3)\cdot(t-1)$$

This tells us that $\text{E-Vals}(H) = \{3, 1\}$. The eigenspaces are

$$\mathcal{E}_H(3) = \operatorname{Null} \overset{3\cdot I_2 - H}{\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}} \qquad \mathcal{E}_H(1) = \operatorname{Null} \overset{1\cdot I_2 - H}{\begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix}}$$

Note that $\mathrm{Col}_2 = i \cdot \mathrm{Col}_1$ in $3 \cdot I_2 - H$ and $\mathrm{Col}_2 = -i \cdot \mathrm{Col}_1$ in $1 \cdot I_2 - H$. These column relations can be re-written as

$$-i \cdot \mathrm{Col}_1 + \mathrm{Col}_2 = \boldsymbol{O} \qquad i \cdot \mathrm{Col}_1 + \mathrm{Col}_2 = \boldsymbol{O}$$

in $3 \cdot I_2 - H$ and $1 \cdot I_2 - H$ respectively. This gives bases

$$\mathcal{E}_H(3) = \operatorname{Span}\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\} \qquad \mathcal{E}_H(1) = \operatorname{Span}\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$$

Here, we have normalized our basis vectors to ensure their orthonormality.

Now, we have a spectral factorization

$$\overset{H}{\begin{bmatrix} 2 & -i \\ i & 2 \end{bmatrix}} = \overset{U}{\frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}} \overset{D}{\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}} \overset{U^*}{\frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}}$$

We then have

$$\begin{aligned} 2 \cdot \exp(H) &= 2 \cdot \overset{U}{\frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}} \overset{D}{\begin{bmatrix} e^3 & \\ & e^1 \end{bmatrix}} \overset{U^*}{\frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}} \\ &= \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} ie^3 & e^3 \\ -ie & e \end{bmatrix} \\ &= \begin{bmatrix} e^3 + e & -ie^3 + ie \\ ie^3 - ie & e^3 + e \end{bmatrix} \end{aligned}$$

Problem 5. Consider the singular value decomposition $A = U\Sigma V^\intercal$ where

$$A = \begin{bmatrix} -1/42 & 1/6 & 1/21 & 5/21 \\ 2/7 & 3/14 & -3/14 & 1/14 \\ -1/42 & 5/21 & -1/6 & -5/42 \\ 1/42 & -1/6 & -1/21 & -5/21 \\ -1/3 & 4/21 & 2/21 & 1/21 \end{bmatrix} \quad U = \frac{1}{\sqrt{7}}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \quad V = \frac{1}{\sqrt{7}}\begin{bmatrix} -2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

Note that A is 5×4 (which means that $A^\intercal A$ is 4×4).

(9 pts) (*a*) $\operatorname{rank}(A) =$ 3, $\operatorname{trace}(U^\intercal U) =$ 3, and $\det(V^\intercal V) =$ 1

(4 pts) (*b*) Only one of the following statements accurately describes the definiteness of $A^\intercal A$. Select this statment.

- ◯ $A^\intercal A$ is indefinite
- ◯ $A^\intercal A$ is positive semidefinite and positive definite
- ◯ the concept of definiteness does not apply to $A^\intercal A$
- √ $A^\intercal A$ is positive semidefinite but not positive definite
- ◯ $A^\intercal A$ is positive definite but not positive semidefinite

(4 pts) (*c*) Only one of the following statements accurately describes the eigenspaces of $A^\intercal A$. Select this statement.

- √ $A^\intercal A$ has one two-dimensional eigenspace and two one-dimensional eigenspaces
- ◯ $A^\intercal A$ has four one-dimensional eigenspaces
- ◯ $A^\intercal A$ has three one-dimensional eigenspaces
- ◯ $A^\intercal A$ has one two-dimensional eigenspace and one one-dimensional eigenspace
- ◯ $A^\intercal A$ has one three-dimensional eigenspace

(10 pts) (*d*) Let $\widehat{\boldsymbol{x}}$ be any solution to the least squares problem associated to $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = \begin{bmatrix} 0 & \sqrt{7} & 0 & 0 & 0 \end{bmatrix}^\intercal$. Calculate $V^*\widehat{\boldsymbol{x}}$. Clearly explain your reasoning to receive credit.

Hint. Start by clearly articulating what system $\widehat{\boldsymbol{x}}$ solves and explain the relevance of SVD to this sytem.

Solution. Swapping in the singular value decomposition $A = U\Sigma V^*$ into the least squares system $A^\intercal A\widehat{x} = A^\intercal \boldsymbol{b}$ gives

$$V\Sigma^2 V^*\widehat{\boldsymbol{x}} = V\Sigma U^*\boldsymbol{b}$$

We know that $V^*V = I_3$ since V has orthonormal columns, so multiplying through by V^* gives

$$\Sigma^2 V^*\widehat{\boldsymbol{x}} = \Sigma U^*\boldsymbol{b}$$

Multiplying by Σ^{-1} gives

$$\Sigma V^*\widehat{\boldsymbol{x}} = U^*\boldsymbol{b}$$

Multiplying again by Σ^{-1} finally gives

$$V^*\boldsymbol{b} = \overbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}^{\Sigma^{-1}} \overbrace{\frac{1}{\sqrt{7}}\begin{bmatrix} 1 & -1 & 0 & -1 & 2 \\ 1 & 2 & 1 & -1 & 0 \\ -1 & 0 & 2 & 1 & 1 \end{bmatrix}}^{U^*} \overbrace{\begin{bmatrix} 0 \\ \sqrt{7} \\ 0 \\ 0 \\ 0 \end{bmatrix}}^{\boldsymbol{b}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$$