Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: ______________________ *Unique ID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

April 18, 2025

- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

Problem 1. Suppose that $A = QR$ where A is 5×4 with $\operatorname{rank}(A) = 2$. Let $P = QQ^\intercal$ and let d be the degree of the characteristic polynomial of P.

(4 pts) (*a*) Q is _______ $\times$ _______ and R is _______ $\times$ _______

(5 pts) (*b*) $d =$ _______ and the coefficient of t^{d-1} in $\chi_P(t)$ is _______

(4 pts) (*c*) $\operatorname{rank}(PA) =$ _______

Problem 2. Let $\boldsymbol{v} \in \mathbb{R}^2$ and let A be the matrix with eigenspaces $\mathcal{E}_A(-2) = \operatorname{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ and $\mathcal{E}_A(1) = \operatorname{Span}\{\boldsymbol{v}\}$.

(4 pts) (*a*) $\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$

(4 pts) (*b*) Suppose $x \neq 0$. Only one of the following choices for the vector $\boldsymbol{v}$ makes the matrix A real-symmetric. Select this vector.

○ $\boldsymbol{v} = \begin{bmatrix}x\\x\end{bmatrix}$ ○ $\boldsymbol{v} = \begin{bmatrix}x\\-x\end{bmatrix}$ ○ $\boldsymbol{v} = \begin{bmatrix}-x\\-x\end{bmatrix}$ ○ $\boldsymbol{v} = \begin{bmatrix}x\\0\end{bmatrix}$ ○ $\boldsymbol{v} = \begin{bmatrix}0\\x\end{bmatrix}$

(10 pts) (*c*) Calculate the matrix-vector product $\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$ assuming that $\boldsymbol{v} = \begin{bmatrix}2\\4\end{bmatrix}$.

Problem 3. This 4×4 matrix A has exactly two eigenvalues. One of the eigenvalues of A is $\lambda_1 = 3$. The geometric and algebraic multiplicities of $\lambda_1 = 3$ satisfy

$$\operatorname{gm}_A(3) = \operatorname{am}_A(3) = 2$$

The other eigenvalue λ_2 is unknown. ***It is known that A is not diagonalizable.***

$$A = \begin{bmatrix} -3 & -41 & -4 & -6 \\ 3 & 24 & 2 & 3 \\ -12 & -86 & -5 & -12 \\ -9 & -63 & -6 & -6 \end{bmatrix}$$

(6 pts) (*a*) $\operatorname{gm}_A(\lambda_2) =$ _______ and $\operatorname{am}_A(\lambda_2) =$ _______

(10 pts) (*b*) Calculate the values of λ_2 and $\det(A)$. Clearly explain your reasoning to receive credit and *fill in the blanks below to make your answers clear.*

The context of this problem allows for the calculation of these numbers with minimal arithmetic, so no partial credit for arithmetic errors will be awarded.

$\lambda_2 =$ _______ and $\det(A) =$ _______

(10 pts) (*c*) There is a basis of $\mathcal{E}_A(3)$ of the form $\{\boldsymbol{v}_1, \boldsymbol{v}_2 = \begin{bmatrix} -2 & 0 & 3 & 0 \end{bmatrix}^\intercal\}$. Find a valid choice for $\boldsymbol{v}_1$ and then use your basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to find an *orthonormal* basis $\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ of $\mathcal{E}_A(3)$. Clearly explain your reasoning to receive credit and *fill in the blank vectors below to make your answer clear.*

$\boldsymbol{q}_1 = \begin{bmatrix} \quad \end{bmatrix}$ and $\boldsymbol{q}_2 = \begin{bmatrix} \quad \end{bmatrix}$

(16 pts) **Problem 4.** Let H be the Hermitian matrix $H = \begin{bmatrix} 2 & -i \\ i & 2 \end{bmatrix}$. Calculate $2 \cdot \exp(H)$. Clearly explain your reasoning and *fill in the blank matrix below to make you answer clear.*

$$2 \cdot \exp(H) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Problem 5. Consider the singular value decomposition $A = U\Sigma V^\intercal$ where

$$A = \begin{bmatrix} -1/42 & 1/6 & 1/21 & 5/21 \\ 2/7 & 3/14 & -3/14 & 1/14 \\ -1/42 & 5/21 & -1/6 & -5/42 \\ 1/42 & -1/6 & -1/21 & -5/21 \\ -1/3 & 4/21 & 2/21 & 1/21 \end{bmatrix} \quad U = \frac{1}{\sqrt{7}}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \quad V = \frac{1}{\sqrt{7}}\begin{bmatrix} -2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

Note that A is 5×4 (which means that $A^\intercal A$ is 4×4).

(9 pts) (*a*) $\operatorname{rank}(A) =$ ________, $\operatorname{trace}(U^\intercal U) =$ ________, and $\det(V^\intercal V) =$ ________

(4 pts) (*b*) Only one of the following statements accurately describes the definiteness of $A^\intercal A$. Select this statment.

○ $A^\intercal A$ is indefinite ○ $A^\intercal A$ is positive semidefinite and positive definite

○ the concept of definiteness does not apply to $A^\intercal A$ ○ $A^\intercal A$ is positive semidefinite but not positive definite

○ $A^\intercal A$ is positive definite but not positive semidefinite

(4 pts) (*c*) Only one of the following statements accurately describes the eigenspaces of $A^\intercal A$. Select this statement.

○ $A^\intercal A$ has one two-dimensional eigenspace and two one-dimensional eigenspaces

○ $A^\intercal A$ has four one-dimensional eigenspaces ○ $A^\intercal A$ has three one-dimensional eigenspaces

○ $A^\intercal A$ has one two-dimensional eigenspace and one one-dimensional eigenspace

○ $A^\intercal A$ has one three-dimensional eigenspace

(10 pts) (*d*) Let $\widehat{\boldsymbol{x}}$ be any solution to the least squares problem associated to $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = \begin{bmatrix} 0 & \sqrt{7} & 0 & 0 & 0 \end{bmatrix}^\intercal$. Calculate $V^*\widehat{\boldsymbol{x}}$. Clearly explain your reasoning to receive credit.
Hint. Start by clearly articulating what system $\widehat{\boldsymbol{x}}$ solves and explain the relevance of SVD to this sytem.