Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: Solutions

Unique ID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

February 6, 2026

- There are 100 points and 7 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. The equation to the right of this paragraph depicts the calculation of a matrix-vector product $A\boldsymbol{x} = \boldsymbol{v}$. Note that the columns of A are unknown and labeled as $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$, that $\boldsymbol{x}$ is the vector $\boldsymbol{x} = \begin{bmatrix}-1 & 2 & 3\end{bmatrix}^\intercal$, and that $\boldsymbol{v}$ is the vector $\boldsymbol{v} = \begin{bmatrix}3 & 2 & -1 & 8\end{bmatrix}^\intercal$.

$$\overset{A}{\begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix}} \overset{\boldsymbol{x}}{\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}} = \overset{\boldsymbol{v}}{\begin{bmatrix} 3 \\ 2 \\ -1 \\ 8 \end{bmatrix}}$$

(2 pts) (*a*) A is ___4___ $\times$ ___3___

(4 pts) (*b*) $\|\boldsymbol{x}\| =$ ___$\sqrt{14}$___ and $\|A\boldsymbol{x}\|^2 =$ ___78___

(4 pts) (*c*) All but one of the following matrix-vector products is guaranteed to equal $\boldsymbol{v}$. Select the matrix-vector product that does *not* equal $\boldsymbol{v}$.

○ $\begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_3 & \boldsymbol{a}_2 \\ | & | & | \end{bmatrix}\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ √ $\begin{bmatrix} | & | & | \\ \boldsymbol{a}_2 & \boldsymbol{a}_1 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} | & | & | \\ \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_1 \\ | & | & | \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ ○ $\begin{bmatrix} | & | & | \\ \boldsymbol{a}_3 & \boldsymbol{a}_1 & \boldsymbol{a}_2 \\ | & | & | \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

Solution. The equation given in the problem is the same as $-\boldsymbol{a}_1 + 2\,\boldsymbol{a}_2 + 3\,\boldsymbol{a}_3 = \boldsymbol{v}$. The four options here are

$$-\boldsymbol{a}_1 + 3\,\boldsymbol{a}_3 + 2\,\boldsymbol{a}_2 \qquad -\boldsymbol{a}_2 + 2\,\boldsymbol{a}_1 + 3\,\boldsymbol{a}_3 \qquad 2\,\boldsymbol{a}_2 + 3\,\boldsymbol{a}_3 - \boldsymbol{a}_1 \qquad 3\,\boldsymbol{a}_3 - \boldsymbol{a}_1 + 2\,\boldsymbol{a}_2$$

The first, third, and fourth linear combinations are the same as $-\boldsymbol{a}_1 + 2\,\boldsymbol{a}_2 + 3\,\boldsymbol{a}_3$, which equals $\boldsymbol{v}$. The second linear combination is not equal to this quantity.

(4 pts) (*d*) Fill-in the missing coordinate to make this matrix-vector product correct: $\begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{v} \\ | & | & | & | \end{bmatrix}\begin{bmatrix} -2 \\ 4 \\ 6 \\ \underline{\;-2\;} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Solution. The equation given in the problem is the same as $-\boldsymbol{a}_1 + 2\,\boldsymbol{a}_2 + 3\,\boldsymbol{a}_3 = \boldsymbol{v}$. If we called the missing coordinate of the given matrix-vector product here c, then we would have

$$\boldsymbol{O} = -2\,\boldsymbol{a}_1 + 4\,\boldsymbol{a}_2 + 6\,\boldsymbol{a}_3 + c\,\boldsymbol{v} = 2(-\boldsymbol{a}_1 + 2\,\boldsymbol{a}_2 + 3\,\boldsymbol{a}_3) + c\,\boldsymbol{v} = 2\boldsymbol{v} + c\,\boldsymbol{v} = (2 + c)\,\boldsymbol{v}$$

This only works for $c = -2$.

(10 pts) (*e*) Let $M = \begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_3 & \boldsymbol{a}_2 & \boldsymbol{a}_1 & \boldsymbol{v} \\ | & | & | & | \end{bmatrix}$. Show that $\boldsymbol{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 8 \end{bmatrix}$ is an eigenvector of M and identify the corresponding eigenvalue λ. Fill in your value of λ in the blank at the bottom of this page for clarity.

Solution. The equation given in the problem is the same as $-\boldsymbol{a}_1 + 2\,\boldsymbol{a}_2 + 3\,\boldsymbol{a}_3 = \boldsymbol{v}$. We are given here that the four columns of M are $\boldsymbol{a}_3$, $\boldsymbol{a}_2$, $\boldsymbol{a}_1$, and $\boldsymbol{v}$.

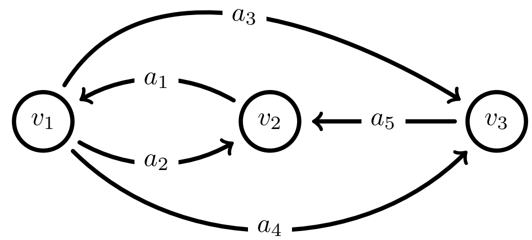
We wish to demonstrate that $M\boldsymbol{v} = \lambda \cdot \boldsymbol{v}$ for some scalar value of λ. To do so, we calculate

$$\overset{M}{\begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_3 & \boldsymbol{a}_2 & \boldsymbol{a}_1 & \boldsymbol{v} \\ | & | & | & | \end{bmatrix}} \overset{\boldsymbol{v}}{\begin{bmatrix} 3 \\ 2 \\ -1 \\ 8 \end{bmatrix}} = 3\,\boldsymbol{a}_3 + 2\,\boldsymbol{a}_2 - \boldsymbol{a}_1 + 8\,\boldsymbol{v} = -\boldsymbol{a}_1 + 2\,\boldsymbol{a}_2 + 3\,\boldsymbol{a}_3 + 8\,\boldsymbol{v} = \boldsymbol{v} + 8\,\boldsymbol{v} = 9\boldsymbol{v}$$

We find here that $M\boldsymbol{v} = -\boldsymbol{v}$, which is $M\boldsymbol{v} = \lambda \cdot \boldsymbol{v}$ with $\lambda = 9$.

$\lambda =$ ___9___

Problem 2. Let A be the incidence matrix of the directed graph G depicted to the right of this sentence.



(2 pts) (*a*) The fourth column of A is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(2 pts) (*b*) $\chi(G) =$ $\underline{\quad -2 \quad}$

(4 pts) (*c*) The notation $\mathbb{R}^x \xrightarrow{A} \mathbb{R}^y$ is valid for $x =$ $\underline{\quad 5 \quad}$ and $y =$ $\underline{\quad 3 \quad}$. The circuit rank of G is $\underline{\quad 3 \quad}$.

(10 pts) **Problem 3.** Supppose that S and K are 2026×2026 matrices where S is symmetric and K is *skew-symmetric* (this means that $K^\intercal = -K$) and let $C = SK - KS$. Show that C is symmetric.

You must clearly justify your reasoning and avoid circular logic to receive credit.

Solution. We are told that S is *symmetric*, which means $S^\intercal = S$. We are also told that $K^\intercal = -K$ and $C = SK - KS$. We wish to show that C is *symmetric*, which means we need to demonstrate that $C^\intercal = C$. To do so, note that

$$\begin{aligned} C^\intercal &= (SK - KS)^\intercal \\ &= (SK)^\intercal - (KS)^\intercal \\ &= K^\intercal S^\intercal - S^\intercal K^\intercal \\ &= (-K)S - S(-K) \\ &= -KS + SK \\ &= SK - KS \\ &= C \end{aligned}$$

We have successfully demonstrated that $C^\intercal = C$, which means that C is indeed symmetric.

(8 pts) **Problem 4.** The fact that the 2026×2026 matrix C from the previous problem is symmetric tells us that some, but not all of the following statements are guaranteed to be true. Select these statements (2pts each).

√ The Gramian of C is equal to C^2. ◯ C must be nonsingular.

√ $\langle C\boldsymbol{v}, \boldsymbol{w}\rangle = \langle \boldsymbol{v}, C\boldsymbol{w}\rangle$ for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^{2026}$ ◯ $\|C\boldsymbol{v}\|^2 = \|\boldsymbol{v}\|$ for any $\boldsymbol{v} \in \mathbb{R}^{2026}$

(10 pts) **Problem 5.** Fill-in the blank next to each of the following matrices with the appropriate notation to indicate the first step called for by the Gauß-Jordan algorithm as articulated in class. You do not need to perform the calculation but you must use correct notation to receive credit. (No partial credit. 2.5pts each)

$\begin{bmatrix} 13 & 19 & 142 & 3957 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ $\underline{\quad \frac{1}{13} \cdot \boldsymbol{r}_1 \to \boldsymbol{r}_1 \quad}$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 27 & 19 & 41 \\ 1 & 0 & 0 \end{bmatrix}$ $\underline{\quad \boldsymbol{r}_1 \leftrightarrow \boldsymbol{r}_4 \quad}$

$\begin{bmatrix} 1 & 5 & 0 & 2 & 8 & 0 \\ 0 & 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ $\underline{\quad \boldsymbol{r}_1 - 8 \cdot \boldsymbol{r}_3 \to \boldsymbol{r}_1 \quad}$

$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ $\underline{\quad \boldsymbol{r}_1 \leftrightarrow \boldsymbol{r}_4 \quad}$

Problem 6. The equation below depicts the product of two 4×4 matrices A and B.

$$\overset{A}{\begin{bmatrix} -5 & 10 & -4 & -2 \\ 0 & 5 & -4 & 4 \\ 3 & -1 & -1 & 5 \\ 3 & -3 & 0 & 4 \end{bmatrix}} \overset{B}{\begin{bmatrix} 12 & -10 & -8 & 26 \\ 0 & 2 & -8 & 8 \\ -9 & 13 & -10 & -5 \\ -9 & 9 & 0 & -15 \end{bmatrix}} = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

It is known that A^{-1} exists and that $A^4 - 3\,A^3 - 3\,A^2 + 11\,A - 6\,I_4 = \boldsymbol{O}$.

(2 pts) (a) All but one of the following vectors is orthogonal to the third row of A. Select this vector.

○ The 1st column of B. ○ The 2nd column of B. √ The 3rd column of B. ○ The 4th column of B.

(3 pts) (b) Only one of the following formulas for A^{-1} is correct. Select this formula.

○ $A^{-1} = B$ ○ $A^{-1} = -6\,B$ ○ $A^{-1} = \frac{1}{6}\left(A^4 - 3\,A^3 - 3\,A^2 + 11\,A\right)$

√ $A^{-1} = \frac{1}{6}\left(A^3 - 3\,A^2 - 3\,A + 11\,I_4\right)$ ○ $A^{-1} = A^3 - 3\,A^2 - 3\,A + 11\,I_4$

(4 pts) (c) If possible, find a scalar c such that $A^{-1} = c \cdot B$ and fill in the blank below for clarity. If it is not possible to find such a value of c, then select "no such c exists.

Clearly explain your reasoning to receive credit.

Solution. The equation given to us is $AB = -6 \cdot I_4$, which can be rearranged as $A\left(-\frac{1}{6}B\right) = I_4$. Since the inverse of a matrix is the only other matrix you can multiply by to produce the identity, this demonstrates that $A^{-1} = -\frac{1}{6}B$. So, our desired value of c is $c = -\frac{1}{6}$.

$c =$ $-\frac{1}{6}$ ○ no such c exists.

(7 pts) (d) Let $\boldsymbol{b} = \begin{bmatrix} 12 & 6 & 0 & 0 \end{bmatrix}^\intercal$. Find the solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$. Fill in the blank below for clarity.

Clearly explain your reasoning to receive credit.

Solution. The matrix A is invertible and $A^{-1} = -\frac{1}{6}B$. The solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = -\frac{1}{6}\overset{B}{\begin{bmatrix} 12 & -10 & -8 & 26 \\ 0 & 2 & -8 & 8 \\ -9 & 13 & -10 & -5 \\ -9 & 9 & 0 & -15 \end{bmatrix}} \overset{\boldsymbol{b}}{\begin{bmatrix} 12 \\ 6 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 12 & -10 & -8 & 26 \\ 0 & 2 & -8 & 8 \\ -9 & 13 & -10 & -5 \\ -9 & 9 & 0 & -15 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -14 \\ -2 \\ 5 \\ 9 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} -14 \\ -2 \\ 5 \\ 9 \end{bmatrix}$$

Problem 7. The data below depicts the row-reduction of a 4×5 matrix A (whose columns are labeled with $\boldsymbol{a}$'s) to $R = \operatorname{rref}(A)$ (whose columns are labeled with $\boldsymbol{c}$'s).

$$\overset{A}{\begin{bmatrix} | & | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 \\ | & | & | & | & | \end{bmatrix}} \xrightarrow{\frac{1}{2}\cdot \boldsymbol{r}_2 \to \boldsymbol{r}_2} \begin{bmatrix} A_1 \end{bmatrix} \xrightarrow[]{\substack{\boldsymbol{r}_1 + m_1 \cdot \boldsymbol{r}_2 \to \boldsymbol{r}_1 \\ \boldsymbol{r}_3 + m_2 \cdot \boldsymbol{r}_2 \to \boldsymbol{r}_3 \\ \boldsymbol{r}_4 + m_3 \cdot \boldsymbol{r}_2 \to \boldsymbol{r}_4}} \begin{bmatrix} A_2 \end{bmatrix} \xrightarrow{\boldsymbol{r}_2 - 2\cdot \boldsymbol{r}_3 \to \boldsymbol{r}_2} \overset{R=\operatorname{rref}(A)}{\begin{bmatrix} | & | & | & | & | \\ \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 & \boldsymbol{c}_4 & \boldsymbol{c}_5 \\ | & | & | & | & | \end{bmatrix}}$$

These operations give $EA = R$ where E is the product of five elementary matrices $E = E_5E_4E_3E_2E_1$.

Note that the scalars used to reduce A_1 to A_2 are notated as m_1, m_2, and m_3 and are unspecified.

(2 pts) (*a*) Each elementary matrix in the product $E = E_5E_4E_3E_2E_1$ is __4__ $\times$ __4__.

(2 pts) (*b*) The row operation that reduces the matrix A_1 back to A is ____ $2 \cdot \boldsymbol{r}_2 \to \boldsymbol{r}_2$ ____.

(4 pts) (*c*) $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

For the rest of this problem, assume that the "column relations" of $R = \operatorname{rref}(A)$ ***are***

$$\boldsymbol{c}_3 = 2\,\boldsymbol{c}_1 - 3\,\boldsymbol{c}_2 \qquad\qquad \boldsymbol{c}_5 = \boldsymbol{c}_2 - \boldsymbol{c}_4$$

(and remember that both A ***and*** R ***are*** 4×5***).***

(6 pts) (*d*) $\operatorname{rank}(R) =$ __3__, $\operatorname{nullity}(R) =$ __2__, and $\operatorname{nullity}(R^\intercal) =$ __1__.

(10 pts) (*e*) It is known that columns $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_4$ of A satisfy

$$\|\boldsymbol{a}_1\| = 1 \qquad \|\boldsymbol{a}_2\| = \sqrt{7} \qquad \langle \boldsymbol{a}_1, \boldsymbol{a}_2\rangle = -1 \qquad \langle \boldsymbol{a}_1, \boldsymbol{a}_4\rangle = -2 \qquad \langle \boldsymbol{a}_2, \boldsymbol{a}_4\rangle = 13$$

Use this information to calculation $\langle \boldsymbol{a}_3, \boldsymbol{a}_5\rangle$. Fill in the blank at the bottom of this page for clarity.

Solution. We know that A and R have the same column relations, so our inner product can be calculated as

$$\begin{aligned} \langle \boldsymbol{a}_3, \boldsymbol{a}_5\rangle &= \langle 2\,\boldsymbol{a}_1 - 3\,\boldsymbol{a}_2, \boldsymbol{a}_2 - \boldsymbol{a}_4\rangle \\ &= 2\,\langle \boldsymbol{a}_1, \boldsymbol{a}_2 - \boldsymbol{a}_4\rangle - 3\,\langle \boldsymbol{a}_2, \boldsymbol{a}_2 - \boldsymbol{a}_4\rangle \\ &= 2\,\langle \boldsymbol{a}_1, \boldsymbol{a}_2\rangle - 2\,\langle \boldsymbol{a}_1, \boldsymbol{a}_4\rangle - 3\,\langle \boldsymbol{a}_2, \boldsymbol{a}_2\rangle + 3\,\langle \boldsymbol{a}_2, \boldsymbol{a}_4\rangle \\ &= 2\,\langle \boldsymbol{a}_1, \boldsymbol{a}_2\rangle - 2\,\langle \boldsymbol{a}_1, \boldsymbol{a}_4\rangle - 3\,\|\boldsymbol{a}_2\|^2 + 3\,\langle \boldsymbol{a}_2, \boldsymbol{a}_4\rangle \\ &= 2\,(-1) - 2\,(-2) - 3\,(\sqrt{7})^2 + 3\,(13) \\ &= -2 + 4 - 21 + 39 \\ &= 20 \end{aligned}$$

$\langle \boldsymbol{a}_3, \boldsymbol{a}_5\rangle =$ __20__