Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: Solutions

Unique ID: ____________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ____________

March 6, 2026

- There are 100 points and 7 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

(4 pts) **Problem 1.** Only one of the following scalars is an eigenvalue of $A = \begin{bmatrix} 7 & 16 \\ -1 & -1 \end{bmatrix}$. Select this scalar.

○ $\lambda = 1$ ○ $\lambda = 2$ ✓ $\lambda = 3$ ○ $\lambda = 4$

Solution. The only singular matrix among $\overset{1\cdot I_2 - A}{\begin{bmatrix} -6 & -16 \\ 1 & 2 \end{bmatrix}}$, $\overset{2\cdot I_2 - A}{\begin{bmatrix} -5 & -16 \\ 1 & 3 \end{bmatrix}}$, $\overset{3\cdot I_2 - A}{\begin{bmatrix} -4 & -16 \\ 1 & 4 \end{bmatrix}}$, and $\overset{4\cdot I_2 - A}{\begin{bmatrix} -3 & -16 \\ 1 & 5 \end{bmatrix}}$ is $3 \cdot I_2 - A$.

(4 pts) **Problem 2.** It is known that $\lambda = 7$ is an eigenvalue of a 3×3 matrix A and that $7 \cdot I_3 - A = \begin{bmatrix} -13 & 20 & -20 \\ -25 & 32 & -32 \\ -9 & 9 & -9 \end{bmatrix}$.
These conditions allow us to infer that exactly one of the following vectors is an eigenvector of A corresponding to the eigenvalue $\lambda = 7$. Select this vector.

○ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ✓ $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(12 pts) **Problem 3.** The number $\lambda = 5$ is an eigenvalue of the matrix M depicted to the right of this paragraph. Use the row reductions called for by the "$PA = LU$ Factorization Algorithm" described in class on the appropriate matrix A to calculate $\operatorname{gm}_M(\lambda)$. To receive credit, you must follow the steps of the reduction algorithm precisely and label your row operations with proper notation. You do not need to calculate the matrices L and P associated with this algorithm. Record your answer in the blank below for clarity.

$$M = \begin{bmatrix} 5 & -4 & -6 & -1 \\ -3 & 2 & 6 & 0 \\ -1 & -9 & -5 & -9 \\ -2 & -6 & -2 & 3 \end{bmatrix}$$

Solution. We need to calculate $\operatorname{gm}_M(\lambda) = \operatorname{nullity}(\lambda \cdot I_4 - M)$, which is

$$\overset{\lambda\cdot I_4 - M}{\begin{bmatrix} 0 & 4 & 6 & 1 \\ 3 & 3 & -6 & 0 \\ 1 & 9 & 10 & 9 \\ 2 & 6 & 2 & 2 \end{bmatrix}} \xrightarrow{\boldsymbol{r}_1 \leftrightarrow \boldsymbol{r}_2} \begin{bmatrix} 3 & 3 & -6 & 0 \\ 0 & 4 & 6 & 1 \\ 1 & 9 & 10 & 9 \\ 2 & 6 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow[\boldsymbol{r}_4 - \frac{2}{3}\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_4]{\boldsymbol{r}_3 - \frac{1}{3}\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_3} \begin{bmatrix} 3 & 3 & -6 & 0 \\ 0 & 4 & 6 & 1 \\ 0 & 8 & 12 & 9 \\ 0 & 4 & 6 & 2 \end{bmatrix}$$

$$\xrightarrow[\boldsymbol{r}_4 - \boldsymbol{r}_2 \to \boldsymbol{r}_4]{\boldsymbol{r}_3 - 2\cdot\boldsymbol{r}_2 \to \boldsymbol{r}_3} \begin{bmatrix} 3 & 3 & -6 & 0 \\ 0 & 4 & 6 & 1 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\boldsymbol{r}_4 - \frac{1}{7}\boldsymbol{r}_3 \to \boldsymbol{r}_4} \begin{bmatrix} 3 & 3 & -6 & 0 \\ 0 & 4 & 6 & 1 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second, and fourth columns of $\lambda \cdot I_4 - M$ are pivot columns and the third is a nonpivot column. This means that $\operatorname{gm}_M(\lambda) = \operatorname{nullity}(\lambda \cdot I_4 - M) = 1$.

$\operatorname{gm}_M(\lambda) =$ __1__

Problem 4. The calculation to the right of this paragraph depicts the reduced row echelon form of an 5×4 matrix A. Note that the columns of A are not specified and labeled as $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$.

$$\operatorname{rref}\overbrace{\begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \\ | & | & | & | \end{bmatrix}}^{A} = \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(4 pts) (a) Only one of the following statements correctly describes the vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$. Select this statement.

◯ $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ is linearly independent √ $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ is linearly dependent

◯ $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ is neither linearly independent nor linearly independent

◯ We cannot establish the independence of $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ without knowing the coordinates of these vectors.

(4 pts) (b) The vectors $\{\begin{bmatrix}1 & 0 & 9 & 0\end{bmatrix}^\intercal, \begin{bmatrix}0 & 1 & -2 & 0\end{bmatrix}^\intercal, \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^\intercal\}$ form a basis exactly one of the following vector spaces. Select this vector space.

◯ $\operatorname{Null}(A)$ ◯ $\operatorname{Col}(A)$ ◯ $\operatorname{Null}(A^\intercal)$ √ $\operatorname{Col}(A^\intercal)$ ◯ none of these

(10 pts) (c) Note that the null space of A is one-dimensional and there are infinitely many vectors $\boldsymbol{v}$ such that $\operatorname{Null}(A) = \operatorname{Span}\{\boldsymbol{v}\}$. However, of all of the vectors $\boldsymbol{v}$ such that $\operatorname{Null}(A) = \operatorname{Span}\{\boldsymbol{v}\}$, only one satisfies $\langle \boldsymbol{v}, \boldsymbol{b}\rangle = -9$ where $\boldsymbol{b} = \begin{bmatrix}1 & 2 & 2 & 11\end{bmatrix}^\intercal$. Find this vector $\boldsymbol{v}$. Clearly explain your reasoning to recieve credit. Record your answer in the blank below for clarity.

Solution. We start by finding a basis of $\operatorname{Null}(A)$. The given $\operatorname{rref}(A)$ tells us that the system $A\boldsymbol{x} = \boldsymbol{O}$ has dependent variables x_1, x_2, x_4 and free variable $x_3 = c_1$. Writing the dependent variables in terms of the free in the general solution vector gives

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9\,c_1 \\ 2\,c_1 \\ c_1 \\ 0 \end{bmatrix} = c_1 \cdot \begin{bmatrix} -9 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

So, every vector $\boldsymbol{v}$ in $\operatorname{Null}(A)$ takes the form $\boldsymbol{v} = c_1 \cdot \begin{bmatrix}-9 & 2 & 1 & 0\end{bmatrix}^\intercal$. The particular value of c_1 satisfying our inner product requirement is then

$$-9 = \langle c_1 \cdot \begin{bmatrix}-9 & 2 & 1 & 0\end{bmatrix}^\intercal, \begin{bmatrix}1 & 2 & 2 & 11\end{bmatrix}^\intercal\rangle = c_1 \cdot \langle \begin{bmatrix}-9 & 2 & 1 & 0\end{bmatrix}^\intercal, \begin{bmatrix}1 & 2 & 2 & 11\end{bmatrix}^\intercal\rangle = c_1 \cdot -3$$

The value of c_1 we are looking for is $c_1 = 3$, which gives $\boldsymbol{v} = \begin{bmatrix}-27 & 6 & 3 & 0\end{bmatrix}^\intercal$.

$$\boldsymbol{v} = \begin{bmatrix} -27 \\ 6 \\ 3 \\ 0 \end{bmatrix}$$

(10 pts) **Problem 5.** Let A be a 2026×2026 matrix that satisfies $A^\intercal A = AA^\intercal$ and suppose that $\boldsymbol{v}$ is in the left null space of A. Show that $\|A\boldsymbol{v}\|^2 = 0$. Clearly explain your reasoning and avoid circular logic to receive credit.

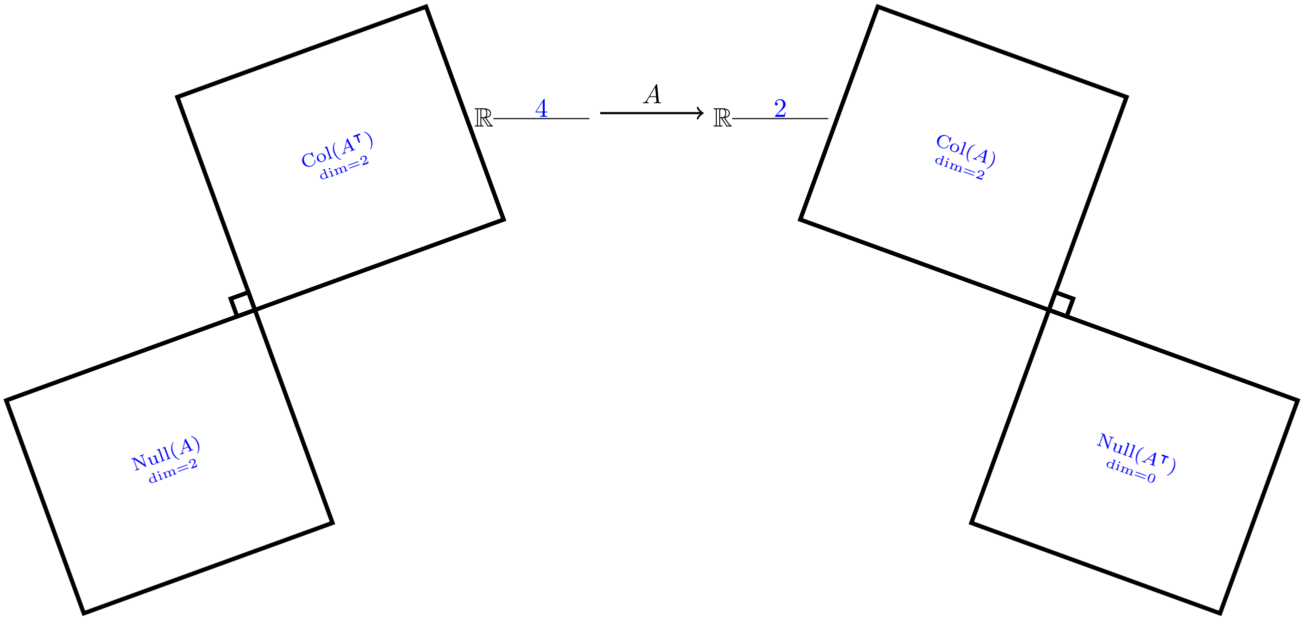
Solution. We are given that $A^\intercal A = AA^\intercal$ and that $\boldsymbol{v}$ is in the left null space of A, which means $A^\intercal \boldsymbol{v} = \boldsymbol{O}$. We wish to demonstrate that $\|A\boldsymbol{v}\|^2 = 0$. To do so, note that

$$\|A\boldsymbol{v}\|^2 = \langle A\boldsymbol{v}, A\boldsymbol{v}\rangle = \langle \boldsymbol{v}, A^\intercal A\boldsymbol{v}\rangle = \langle \boldsymbol{v}, AA^\intercal \boldsymbol{v}\rangle = \langle \boldsymbol{v}, A\boldsymbol{O}\rangle = 0$$

Problem 6. Let $V \subset \mathbb{R}^4$ be the vector space of solutions to the system to the right of this paragraph and let A be the coefficient matrix of this system.

$$\begin{array}{rcrcrcrcl} 2\,x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0 \\ -\,x_1 & - & x_2 & + & 2\,x_3 & + & x_4 & = & 0 \end{array}$$

(10 pts) (a) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(12 pts) (b) Find the projection of $\boldsymbol{b} = \begin{bmatrix} 0 & 7 & 0 & 0 \end{bmatrix}^\intercal$ onto $V^\perp$. Clearly explain your reasoning to receive credit. Fill in the blank vector at the bottom of the page for clarity.

Solution. We wish to calculate the projection of $\boldsymbol{b}$ onto $V^\perp = \operatorname{Null}(A)^\perp = \operatorname{Col}(A^\intercal)$ where $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & -1 & 2 & 1 \end{bmatrix}$. The picture indicates that $\dim(V^\perp) = \dim \operatorname{Col}(A^\intercal) = 2$, which means we can simply use the two rows of A to form a basis of $V^\perp$. The projection matrix onto $V^\perp$ is then $X(X^\intercal X)^{-1}X^\intercal$ where

$$X = \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad X^\intercal X = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

Our projection is then

$$\overset{X}{\begin{bmatrix} 2 & -1 \\ 1 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}} \overset{(X^\intercal X)^{-1}}{\begin{bmatrix} 1/7 & 0 \\ 0 & 1/7 \end{bmatrix}} \overset{X^\intercal}{\begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & -1 & 2 & 1 \end{bmatrix}} \overset{\boldsymbol{b}}{\begin{bmatrix} 0 \\ 7 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/7 & 0 \\ 0 & 1/7 \end{bmatrix} \begin{bmatrix} 7 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

projection of $\boldsymbol{b}$ onto $V^\perp$ is $\begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

Problem 7. The matrix P to the right of this paragraph is the projection matrix onto a vector space $V \subset \mathbb{R}^4$. Throughout this problem we will use the symbol Q to refer to the projection matrix onto $V^\perp$. ***Do not ignore the factor of*** $^1/_{13}$ ***used to define*** P ***(for example, the*** $(2,3)$ ***entry of*** P ***is*** $-^4/_{13}$***).***

$$P = \frac{1}{13}\begin{bmatrix} 12 & 2 & 2 & 2 \\ 2 & 9 & -4 & -4 \\ 2 & -4 & 9 & -4 \\ 2 & -4 & -4 & 9 \end{bmatrix}$$

(3 pts) (*a*) $\dim(V) =$ __3__

(6 pts) (*b*) Then the $(3,2)$ entry of Q is __$^4/_{13}$__ and the $(4,4)$ entry of Q is __$^4/_{13}$__.

(10 pts) (*c*) Let $\boldsymbol{v} = \begin{bmatrix} 6 & 1 & 1 & 1 \end{bmatrix}^\intercal$ and $\boldsymbol{w} = \begin{bmatrix} -1 & 2 & 2 & 2 \end{bmatrix}^\intercal$. It is known that $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in V^\perp$. Use this information to calculate $(P^{2026} + 4\,P^2 - P - I_4)(\boldsymbol{v} + \boldsymbol{w})$. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

Solution. We are given that $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in V^\perp$. Since P is the projection matrix onto V, this means that $P\boldsymbol{v} = \boldsymbol{v}$ and $P\boldsymbol{w} = \boldsymbol{O}$. Additionally, from class we know that P is *idempotent* which means $P^2 = P$ and, consequently, $P^n = P$ for any $n \geq 1$. This allows us to simplify our calculation to

$$\begin{aligned}
(P^{2026} + 4\,P^2 - P - I_4)(\boldsymbol{v} + \boldsymbol{w}) &= (P + 4\,P - P - I_4)(\boldsymbol{v} + \boldsymbol{w}) \\
&= (4\,P - I_4)(\boldsymbol{v} + \boldsymbol{w}) \\
&= (4\,P - I_4)\boldsymbol{v} + (4\,P - I_4)\boldsymbol{w} \\
&= 4\,P\boldsymbol{v} - I_4\boldsymbol{v} + 4\,P\boldsymbol{w} - I_4\boldsymbol{w} \\
&= 4 \cdot \boldsymbol{v} - \boldsymbol{v} + 4 \cdot \boldsymbol{O} - \boldsymbol{w} \\
&= 3 \cdot \boldsymbol{v} - \boldsymbol{w} \\
&= \begin{bmatrix} 18 \\ 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 2 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 19 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

$$(P^{2026} + 4\,P^2 - P - I_4)(\boldsymbol{v} + \boldsymbol{w}) = \begin{bmatrix} 19 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

For the rest of this problem, let A ***be a matrix such that*** $V = \operatorname{Col}(A)$***.***

(3 pts) (*d*) The number of rows of A is __4__.

(8 pts) (*e*) Let $\boldsymbol{b} = \begin{bmatrix} 13 & 0 & 0 & 0 \end{bmatrix}^\intercal$. It is known that the system $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent. Calculate the *error* E associated with replacing $A\boldsymbol{x} = \boldsymbol{b}$ with the system $A^\intercal A\widehat{\boldsymbol{x}} = A^\intercal\boldsymbol{b}$. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

Solution. This is asking for the error in solving the least squares problem associated to $A\boldsymbol{x} = \boldsymbol{b}$, which is $E = \|\boldsymbol{b} - A\widehat{\boldsymbol{x}}\|^2 = \|\boldsymbol{b} - P\boldsymbol{b}\|^2$ where P is the projection matrix onto $\operatorname{Col}(A)$. We are given precisely this projection matrix, so the relevant calculations here are

$$\frac{1}{13}\begin{bmatrix} 12 & 2 & 2 & 2 \\ 2 & 9 & -4 & -4 \\ 2 & -4 & 9 & -4 \\ 2 & -4 & -4 & 9 \end{bmatrix}\begin{bmatrix} 13 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 & 2 & 2 & 2 \\ 2 & 9 & -4 & -4 \\ 2 & -4 & 9 & -4 \\ 2 & -4 & -4 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ 2 \\ 2 \end{bmatrix} \qquad E = \left\| \begin{bmatrix} 13 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 12 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} 1 \\ -2 \\ -2 \\ -2 \end{bmatrix} \right\|^2 = 13$$

$E =$ __13__