DUKE UNIVERSITY

MATH 218D-2

MATRICES AND VECTORS

Exam II

Name: ______________________ *Unique ID:* ______________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

March 6, 2026

- There are 100 points and 7 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

(4 pts) **Problem 1.** Only one of the following scalars is an eigenvalue of $A = \begin{bmatrix} 7 & 16 \\ -1 & -1 \end{bmatrix}$. Select this scalar.

○ $\lambda = 1$ ○ $\lambda = 2$ ○ $\lambda = 3$ ○ $\lambda = 4$

(4 pts) **Problem 2.** It is known that $\lambda = 7$ is an eigenvalue of a 3×3 matrix A and that $7 \cdot I_3 - A = \begin{bmatrix} -13 & 20 & -20 \\ -25 & 32 & -32 \\ -9 & 9 & -9 \end{bmatrix}$. These conditions allow us to infer that exactly one of the following vectors is an eigenvector of A corresponding to the eigenvalue $\lambda = 7$. Select this vector.

○ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(12 pts) **Problem 3.** The number $\lambda = 5$ is an eigenvalue of the matrix M depicted to the right of this paragraph. Use the row reductions called for by the "$PA = LU$ Factorization Algorithm" described in class on the appropriate matrix A to calculate $\mathrm{gm}_M(\lambda)$. To receive credit, you must follow the steps of the reduction algorithm precisely and label your row operations with proper notation. You do not need to calculate the matrices L and P associated with this algorithm. Record your answer in the blank below for clarity.

$$M = \begin{bmatrix} 5 & -4 & -6 & -1 \\ -3 & 2 & 6 & 0 \\ -1 & -9 & -5 & -9 \\ -2 & -6 & -2 & 3 \end{bmatrix}$$

$\mathrm{gm}_M(\lambda) =$ ________

Problem 4. The calculation to the right of this paragraph depicts the reduced row echelon form of an 5×4 matrix A. Note that the columns of A are not specified and labeled as $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$.

$$\operatorname{rref}\overbrace{\begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \\ | & | & | & | \end{bmatrix}}^{A} = \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(4 pts) (*a*) Only one of the following statements correctly describes the vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$. Select this statement.

○ $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ is linearly independent ○ $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ is linearly dependent

○ $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ is neither linearly independent nor linearly independent

○ We cannot establish the independence of $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ without knowing the coordinates of these vectors.

(4 pts) (*b*) The vectors $\{\begin{bmatrix}1 & 0 & 9 & 0\end{bmatrix}^\intercal, \begin{bmatrix}0 & 1 & -2 & 0\end{bmatrix}^\intercal, \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^\intercal\}$ form a basis exactly one of the following vector spaces. Select this vector space.

○ $\operatorname{Null}(A)$ ○ $\operatorname{Col}(A)$ ○ $\operatorname{Null}(A^\intercal)$ ○ $\operatorname{Col}(A^\intercal)$ ○ none of these

(10 pts) (*c*) Note that the null space of A is one-dimensional and there are infinitely many vectors $\boldsymbol{v}$ such that $\operatorname{Null}(A) = \operatorname{Span}\{\boldsymbol{v}\}$. However, of all of the vectors $\boldsymbol{v}$ such that $\operatorname{Null}(A) = \operatorname{Span}\{\boldsymbol{v}\}$, only one satisfies $\langle \boldsymbol{v}, \boldsymbol{b} \rangle = -9$ where $\boldsymbol{b} = \begin{bmatrix}1 & 2 & 2 & 11\end{bmatrix}^\intercal$. Find this vector $\boldsymbol{v}$. Clearly explain your reasoning to recieve credit. Record your answer in the blank below for clarity.

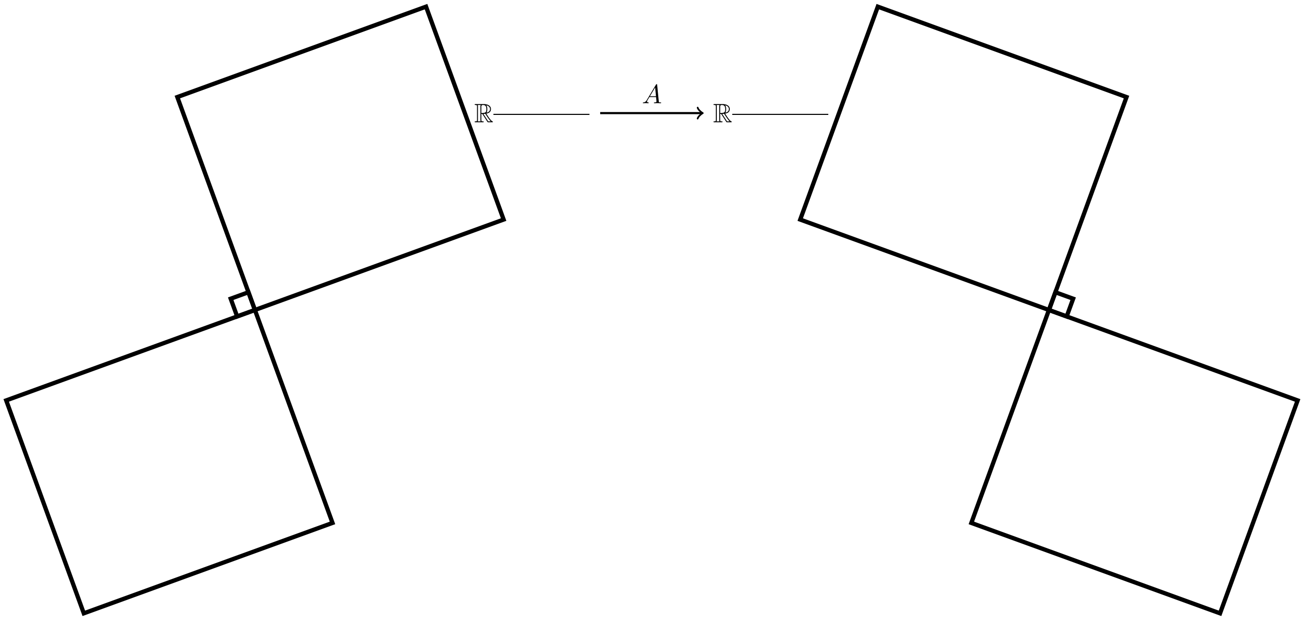
$$\boldsymbol{v} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

(10 pts) **Problem 5.** Let A be a 2026×2026 matrix that satisfies $A^\intercal A = AA^\intercal$ and suppose that $\boldsymbol{v}$ is in the left null space of A. Show that $\|A\boldsymbol{v}\|^2 = 0$. Clearly explain your reasoning and avoid circular logic to receive credit.

Problem 6. Let $V \subset \mathbb{R}^4$ be the vector space of solutions to the system to the right of this paragraph and let A be the coefficient matrix of this system.

$$\begin{array}{rcrcrcrcl} 2\,x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0 \\ -\,x_1 & - & x_2 & + & 2\,x_3 & + & x_4 & = & 0 \end{array}$$

(10 pts) (a) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(12 pts) (b) Find the projection of $\boldsymbol{b} = \begin{bmatrix} 0 & 7 & 0 & 0 \end{bmatrix}^\intercal$ onto $V^\perp$. Clearly explain your reasoning to receive credit. Fill in the blank vector at the bottom of the page for clarity.

projection of $\boldsymbol{b}$ onto $V^\perp$ is $\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$

Problem 7. The matrix P to the right of this paragraph is the projection matrix onto a vector space $V \subset \mathbb{R}^4$. Throughout this problem we will use the symbol Q to refer to the projection matrix onto $V^\perp$. ***Do not ignore the factor of*** $^1/_{13}$ ***used to define*** P ***(for example, the*** $(2,3)$ ***entry of*** P ***is*** $-^4/_{13}$***).***

$$P = \frac{1}{13}\begin{bmatrix} 12 & 2 & 2 & 2 \\ 2 & 9 & -4 & -4 \\ 2 & -4 & 9 & -4 \\ 2 & -4 & -4 & 9 \end{bmatrix}$$

(3 pts) (a) $\dim(V) =$ _______

(6 pts) (b) Then the $(3,2)$ entry of Q is _______ and the $(4,4)$ entry of Q is _______.

(10 pts) (c) Let $\boldsymbol{v} = \begin{bmatrix} 6 & 1 & 1 & 1 \end{bmatrix}^\intercal$ and $\boldsymbol{w} = \begin{bmatrix} -1 & 2 & 2 & 2 \end{bmatrix}^\intercal$. It is known that $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in V^\perp$. Use this information to calculate $(P^{2026} + 4\,P^2 - P - I_4)(\boldsymbol{v} + \boldsymbol{w})$. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

$$(P^{2026} + 4\,P^2 - P - I_4)(\boldsymbol{v} + \boldsymbol{w}) = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

For the rest of this problem, let A ***be a matrix such that*** $V = \operatorname{Col}(A)$***.***

(3 pts) (d) The number of rows of A is _______.

(8 pts) (e) Let $\boldsymbol{b} = \begin{bmatrix} 13 & 0 & 0 & 0 \end{bmatrix}^\intercal$. It is known that the system $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent. Calculate the *error* E associated with replacing $A\boldsymbol{x} = \boldsymbol{b}$ with the system $A^\intercal A\widehat{\boldsymbol{x}} = A^\intercal\boldsymbol{b}$. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

$E =$ _______